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## THE APRIL MEETING IN NEW YORK

The three hundred thirtieth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 10–11, 1936. The attendance included the following one hundred eighty-four members of the Society:

C. R. Adams, R. P. Agnew, E. S. Akeley, R. L. Anderson, R. C. Archibald, Reinhold Baer, F. E. Baker, M. F. Becker, Felix Bernstein, Garrett Birkhoff, Gertrude Blanch, L. M. Blumenthal, R. P. Boas, H. F. Bohnenblust, Samuel Borofsky, J. W. Bower, C. B. Boyer, H. W. Brinkmann, J. H. Bushey, Jewell Hughes Bushey, R. H. Cameron, B. H. Camp, E. W. Cannon, Leonard Carlitz, E. W. Chittenden, W. W. S. Claytor, W. B. Coleman, George Comenetz, E. G. H. Comfort, Richard Courant, A. R. Crathorne, E. L. Culbreth, H. B. Curry, E. H. Cutler, L. S. Dederick, J. L. Doob, Jesse Douglas, Arnold Dresden, Nelson Dunford, Aaron Fialkow, T. S. Fiske, W. B. Fite, M. M. Flood, Tomlinson Fort, M. C. Foster, R. M. Foster, Philip Franklin, T. C. Fry, H. M. Gehman, J. J. Gergen, B. P. Gill, W. C. Graustein, L. M. Graves, M. C. Gray, C. C. Grove, K. W. Halbert, Marshall Hall, H. J. Hamilton, E. R. Hedrick, Robert Henderson, M. R. Hestenes, E. H. C. Hildebrandt, J. D. Hill, Einar Hille, Lulu Hofmann, T. R. Hollcroft, Charles Hopkins, G. M. Hopper, S. E. Hotelling, E. M. Hull, S. A. Joffe, R. A. Johnson, B. W. Jones, I. N. Kagno, E. R. van Kampen, Nathan Kaplan, Edward Kasner, L. S. Kennison, J. R. Kline, M. S. Knebelman, T. L. Koehler, H. L. Krall, Solomon Kullback, K. W. Lamson, R. E. Langer, V. V. Latshaw, Solomon Lefschetz, D. H. Lehmer, Marie Litzinger, W. R. Longley, E. R. Lorch, L. L. Lowenstein, N. H. McCoy, L. A. MacColl, Saunders MacLane, N. B. MacLean, H. M. MacNeille, H. F. MacNeish, R. S. Martin, W. T. Martin, A. E. Meder, G. M. Merriman, A. N. Milgram, E. W. Miller, E. C. Molina, Deane Montgomery, L. T. Moore, D. S. Morse, Marston Morse, S. B. Myers, D. S. Nathan, C. A. Nelson, John von Neumann, E. P. Northrop, C. O. Oakley, Alta Odoms, Oystein Ore, E. R. Ott, F. W. Owens, H. B. Owens, C. K. Payne, T. S. Peterson, H. B. Phillips, A. E. Pitcher, E. L. Post, G. B. Price, R. G. Putnam, H. A. Rademacher, W. C. Randels, Robert Rawhouser, M. S. Rees, Moses Richardson, R. G. D. Richardson, John Riordan, J. H. Roberts, R. M. Robinson, S. L. Robinson, S. G. Roth, H. A. Ruger, I. J. Schoenberg, Wladimir Seidel, Stefan Serghiesco, I. M. Sheffer, Jacob Sherman, Max Shiffman, J. A. Shohat, R. C. Shook, C. G. Shover, L. G. Simons, James Singer, Abraham Sinkov, L. L. Smail, P. A. Smith, N. E. Steenrod, M. H. Stone, D. J. Struik, J. D. Tamarkin, J. M. Thomas, T. Y. Thomas, E. W. Titt, A. W. Tucker, W. S. Turpin, H. E. Vaughan, Oswald Veblen, G. P. Wadsworth, J. F. Wardwell, S. E. Warschawski, C. W. Watkeys, G. C. Webber, Albert Wertheimer, A. P. Wheeler, H. S. White, Hassler Whitney, G. T. Whyburn, S. S. Wilks, W. A. Wilson, E. W. Wilson, Clement Winston, Aurel Wintner, H. P. Wirth, W. H. Wise, H. N. Wright, Leo Zippin, Max Zorn.

The meeting opened on Friday morning with two sectional sessions, one for Algebra and Analysis and one for Geometry.



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Friday afternoon was devoted to a symposium on *Mathematical statistics* at which the following addresses were presented: *Methods of obtaining probability distributions*, by Professor B. H. Camp; *The characterization of a distribution function through its moments*, by Professor J. A. Shohat; *The structure and sampling theory of certain generalized likelihood test criteria*, by Dr. S. S. Wilks. Because of illness Professor Harold Hotelling was unable to present his address, *The method of maximum likelihood*, which was to have been a part of this symposium.

Saturday morning there were two sectional sessions, one for Number Theory and Group Theory and one for Analysis. These sessions were followed by a general session at which Professor C. Kuratowski gave an address entitled *On a topological method of proving existential theorems*.

A meeting of the Council was held at 5:00 P.M. on Friday in the Faculty Club of Columbia University. The election of the following persons to membership in the Society was announced by the Secretary:

Professor Evelyn Niemann Akeley, Skidmore College, Saratoga Springs, N. Y.;
Mr. Martin Collett Bailey, State College, Dover, Del.;
Mr. Clair Eugene Carey, Clarion State Teachers College, Clarion, Pa.;
Mrs. Natividad Rosado Javier, Silliman University, Danaguete, P. I.;
Professor Otto Laporte, University of Michigan;
Dr. Walter Leighton, Harvard University;
Professor Alejandro Melchor, University of the Philippines;
Mr. John Leo Nagle, National Park Service, Washington, D. C.;
Dr. Edwin N. Oberg, Rochester Junior College, Rochester, Minn.;
Mr. Arthur P. O'Mara, Loyola University, Chicago, Ill.;
Professor Karl B. Patterson, Duke University;
Mr. Paul Milton Pepper, University of Cincinnati;
Dr. Raphael Mitchel Robinson, Brown University;
Mr. Walter Edwin Sewell, Harvard University;
Mr. Norman Earl Steenrod, Princeton University;
Dr. John Murray Thompson, University of California, Berkeley;
Professor Albert Potter Wills, Columbia University.

As nominees of the Department of Mathematics of the University of Chicago:

Mr. Carl Herbert Denbow, University of Chicago;
Mr. Nathan Abraham Moscovitch, University of Chicago;
Mr. Malcolm F. Smiley, University of Chicago;
Mr. Frederick Albert Valentine, University of Chicago.

As nominee of Wellesley College:

Miss Mary Dean Clement, Ward-Belmont School, Nashville, Tenn.

It was announced that the following person had entered the Society under the reciprocity agreement:

Dr. Reinhold Baer, Institute for Advanced Study.

The following appointments by President Lefschetz were announced: as representative of the Society at the inauguration of Fred Garrigus Holloway as president of Western Maryland College, April 25, 1936, Professor Abraham Cohen; as representative of the Society on the sectional committee of the American Standards Association, Professor A. A. Bennett; as representatives of the Society to the International Congress of Mathematicians at Oslo, Professors G. D. Birkhoff, H. F. Blichfeldt, L. P. Eisenhart, Solomon Lefschetz, Marston Morse, Virgil Snyder, Oswald Veblen, and Norbert Wiener; as committee on arrangements for the 1936 Annual Meeting, Professors J. M. Thomas (chairman), W. W. Elliott, J. R. Kline, J. W. Lasley, and J. H. Roberts.

It was reported that the invitation of Pennsylvania State College to hold the Summer Meeting of 1937 at that institution had been accepted. Other meetings of the Society in 1937 were announced as follows: New York, February 20, March 26–27; Chicago, April 9–10.

Titles and cross references to the abstracts of the papers (other than symposium addresses) read at this meeting follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 18 were read before the Section for Algebra and Analysis, Professors R. E. Langer and H. B. Phillips presiding; those numbered 19 to 29 before the Section for Geometry, Professor E. W. Chittenden presiding; those numbered 30 to 36 before the Section for Number Theory and Group Theory, Professor T. R. Hollcroft presiding; those numbered 37 to 43 before the Saturday morning Section for Analysis, Professor L. M. Graves presiding; and those numbered 44 to 61 at the general session, Professor Solomon Lefschetz presiding. Mr. E. C. Molina presided at the Symposium. Dr. R. M. Robinson was introduced by Professor J. D. Tamarkin, Mr. J. J. DeCicco by Professor Edward Kasner, and Mr. N. E. Steenrod by Professor Solomon Lefschetz.

1. *Note on some equations without affect*, by Dr. Saunders MacLane. (Abstract No. 42-5-196.)

2. *On products of successive terms of an arithmetic series*, by Mr. Benjamin Rosenbaum. (Abstract No. 42-5-206-*t*.)

3. *A characterization of a class of polynomials*, by Professor I. M. Sheffer. (Abstract No. 42-5-210.)

4. *A new class of transcendental numbers*, Professor Philip Franklin. (Abstract No. 42-5-183.)

5. *On certain equations in a relative-cyclic field*, by Professor Leonard Carlitz. (Abstract No. 42-5-177.)

6. *Products of methods of summability*, by Professor R. P. Agnew. (Abstract No. 42-5-175.)

7. *Bloch functions*, by Dr. R. M. Robinson. (Abstract No. 42-5-205.)

8. *Rational tensor functions of the coefficients of a pair of quadratic differential forms and their first derivatives* (preliminary report), by Professor E. S. Akeley. (Abstract No. 42-5-226.)

9. *Some theorems on Fourier transforms and conjugate trigonometric integrals*, by Mr. R. P. Boas. (Abstract No. 42-3-114.)

10. *On the summability of Fourier series*, by Dr. W. C. Randels. (Abstract No. 42-5-201.)

11. *On the determination of earth conductivity from observed surface potentials*, by Professor R. E. Langer. (Abstract No. 42-5-194.)

12. *On the uniqueness of invariant Lebesgue measures*, by Professor John von Neumann. (Abstract No. 42-5-237.)

13. *On expansions in terms of a certain general class of functions*, by Dr. W. T. Martin. (Abstract No. 42-5-197-*t*.)

14. *On some hermitian forms associated with two given curves of the complex plane*, by Professor Gabriel Szegö. (Abstract No. 42-5-214-*t*.)

15. *On Phragmén-Lindelöf's principle*, by Dr. L. V. Ahlfors. (Abstract No. 42-5-225-*t*.)

16. *On the completing of a Hausdorff space*, by Professor L. M. Graves. (Abstract No. 42-5-185-*t*.)

17. *On the completeness of Lambert functions*. II, by Professors Einar Hille and Otto Szász. (Abstract No. 42-5-229-*t*.)

18. *Direct decompositions*, by Professor Oystein Ore. (Abstract No. 42-5-200-*t*.)

19. *The triangulation of surfaces and the Heawood color formula*, by Mr. I. N. Kagno. (Abstract No. 42-5-189.)

20. *Relations among characteristics of associated linear systems*, by Professor T. R. Hollcroft. (Abstract No. 42-5-187.)

21. *Non-separating transformations*, by Dr. J. F. Wardwell. (Abstract No. 42-5-218.)

22. *Certain extensions of some separation theorems*, by Mr. A. N. Milgram. (Abstract No. 42-5-198.)

23. *On the convergence of manifolds and irreducible membranes*, by Dr. H. E. Vaughan. (Abstract No. 42-5-217.)

24. *Transformations of reference systems in the space-time of Page*, by Professor H. T. Engstrom and Dr. Max Zorn. (Abstract No. 42-5-181.)

25. *Einstein hypersurfaces of a euclidean space*, by Dr. Nathan Kaplan. (Abstract No. 42-3-115.)

26. *General harmonic transformations*, by Professor Edward Kasner. (Abstract No. 42-5-191.)

27. *Geometry of whirl series*, by Mr. J. J. DeCicco. (Abstract No. 42-5-178.)

28. *The maps of a 4-complex into a 2-sphere*, by Professor Hassler Whitney. (Abstract No. 42-5-220.)

29. *On the behavior of a conformal mapping at a cusp*, by Dr. S. E. Warschawski. (Abstract No. 42-5-219.)

30. *On the fundamental group of a certain class of plane algebraic curves*, by Mr. W. S. Turpin. (Abstract No. 42-5-216.)

31. *Concerning groups of order $p^m$ whose operations are of order $\leqq p$* (preliminary report), by Dr. Charles Hopkins. (Abstract No. 42-5-188.)

32. *Universal homology groups*, by Mr. N. E. Steenrod. (Abstract No. 42-5-213.)

33. *Continuous free groups*, by Mr. Garrett Birkhoff. (Abstract No. 42-5-228.)

34. *On a perfect group of order* 10,752, by Dr. Abraham Sinkov. (Abstract No. 42-5-212.)

35. *Number theory in a hypercomplex system*, by Dr. C. G. Shover. (Abstract No. 42-5-211.)

36. *A number theory problem concerning regular simplexes*, by Professor I. J. Schoenberg. (Abstract No. 42-5-207.)

37. *On the order of growth of univalent functions*, by Dr. Wladimir Seidel. (Abstract No. 42-5-209.)

38. *On a property of families of sets*, by Dr. E. W. Miller. (Abstract No. 42-5-199.)

39. *Summability of double Fourier series*, by Professor J. J. Gergen. (Abstract No. 42-5-184.)

40. *Singular quadratic functionals*, by Professor Marston Morse and Dr. Walter Leighton. (Abstract No. 42-3-116-*t*.)

41. *Functional topology and abstract variational theory*, by Professor Marston Morse. (Abstract No. 42-5-230.)

42. *Minimax principle in the calculus of variations*, by Dr. M. R. Hestenes. (Abstract No. 42-5-186.)

43. *The space (BV) of functions of bounded variation*, by Professor C. R. Adams. (Abstract No. 42-5-174.)

44. *Tensors associated with a pair of quadratic differential forms*, by Professor E. S. Akeley. (Abstract No. 42-5-227-*t*.)

45. *A theorem in the theory of numbers as applied to regular polygons*, by Professor W. D. Cairns. (Abstract No. 42-5-176-*t*.)

46. *Remarks on a preceding paper of J. A. Clarkson*, by Mr. Nelson Dunford and Mr. Anthony Morse. (Abstract No. 42-5-179-*t*.)

47. *An involutorial transformation determined by three pencils of planes*, by Professor L. A. Dye. (Abstract No. 42-5-180-*t*.)

48. *Initial motion of fields of force*, by Mr. Aaron Fialkow. (Abstract No. 42-5-182-*t*.)

49. *A unicity theorem for ordinary differential equations and a postulate system for Lie groups*, by Dr. E. R. van Kampen. (Abstract No. 42-5-190-*t*.)

50. *A combinatorial condition for planar graphs*, by Dr. Saunders MacLane. (Abstract No. 42-5-195-*t*.)

51. *On the order of the coefficients of a schlicht function*, by Dr. M. S. Robertson. (Abstract No. 42-5-202-*t*.)

52. *On the summability by positive typical means of sequences* $f(n\theta)$, by Dr. M. S. Robertson. (Abstract No. 42-5-203-*t*.)

53. *Concerning special centers of projection for an algebraic space branch*, by Mr. W. S. Turpin. (Abstract No. 42-5-215-*t*.)

54. *On the principles of d'Alembert, Jourdain, and Gauss*, by Dr. Max Zorn. (Abstract No. 42-5-224-*t*.)

55. *Absolute full reducibility of semi-simple distributive linear algebras*, by Dr. Max Zorn. (Abstract No. 42-5-221-*t*.)

56. *On a formal decomposition theorem for algebras*, by Dr. Max Zorn. (Abstract No. 42-5-223-*t*.)

57. *Galois theory based on the lemmata of Artin and Baer*, by Dr. Max Zorn. (Abstract No. 42-5-222-*t*.)

58. *On certain two-point expansions of integral functions of exponential type*, by Professor I. J. Schoenberg. (Abstract No. 42-5-208-*t*.)

59. *On derivatives of orthogonal polynomials*, by Dr. H. L. Krall. (Abstract No. 42-5-192-*t*.)

60. *On higher derivatives of orthogonal polynomials*, by Dr. H. L. Krall. (Abstract No. 42-5-193-*t*.)

61. *A lacunary function with branch point at the origin*, by Mr. L. B. Robinson. (Abstract No. 42-5-204-*t*.)

J. R. KLINE,
*Associate Secretary*

## THE APRIL MEETING IN CHICAGO

The three hundred thirty-first meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10–11, 1936. There were about one hundred thirty-five persons in attendance among whom were the following ninety-two members of the Society:

A. A. Albert, G. E. Albert, M. R. Anderson, S. F. Barber, R. H. Bardell, I. A. Barnett, Walter Bartky, E. F. Beckenbach, M. M. Beenken, Theodore Bennett, H. R. Beveridge, G. A. Bliss, Henry Blumberg, H. R. Brahana, R. S. Burington, L. E. Bush, W. H. Bussey, Max Coral, J. J. Corliss, D. R. Curtiss, W. M. Davis, W. W. Denton, L. E. Dickson, D. M. Dribin, E. B. Escott, H. P. Evans, H. S. Everett, H. H. Goldstine, L. W. Griffiths, V. G. Grove, V. C. Harris, W. L. Hart, O. C. Hazlett, D. L. Holl, Ralph Hull, M. H. Ingraham, Dunham Jackson, R. L. Jeffery, E. D. Jenkins, Fritz John, P. W. Ketchum, W. S. Kimball, J. M. Kinney, S. C. Kleene, W. C. Krathwohl, A. C. Ladner, Cornelius Lanczos, C. G. Latimer, C. A. Lester, M. I. Logsdon, A. C. Lunn, C. C. MacDuffee, W. D. MacMillan, Morris Marden, A. E. May, J. R. Mayor, L. E. Mehlenbacher, U. G. Mitchell, D. C. Morrow, E. J. Moulton, A. L. Nelson, E. A. Nordhaus, Rufus Oldenburger, G. A. Parkinson, E. D. Pepper, Tibor Radó, G. Y. Rainich, R. B. Rasmusen, W. T. Reid, P. G. Robinson, A. E. Ross, W. E. Roth, N. E. Rutt, R. G. Sanger, Henry Scheffé, Henry Schultz, Nathan Schwid, H. A. Simmons, Burke Smith, E. S. Sokolnikoff, I. S. Sokolnikoff, G. G. Speeker, R. C. Stephens, E. B. Stouffer, Gabriel Szegö, H. P. Thielman, R. W. Wagner, K. W. Wegner, O. L. Wheeler, E. P. Wiggin, F. B. Wiley, F. E. Wood.

The meeting opened on Friday morning with two sectional sessions, one on Algebra and one on Analysis and Geometry. The Symposium Lecture was delivered Friday afternoon by Professor Tibor Radó of Ohio State University on *Some geometrical applications of conformal mapping*. Saturday morning was devoted to a general session.

The dinner which was held at Judson Court on Friday was attended by about one hundred mathematicians and their guests. Dean E. B. Stouffer of the University of Kansas acting as toastmaster invited those present to attend the meeting of the Society at Lawrence, Kansas, next November. He then called upon Professor R. L. Jeffery of Acadia University who is teaching this year at the University of Wisconsin, Dr. E. F. Beckenbach of Rice Institute, and Professor D. R. Curtiss of Northwestern University.

Friday morning Professor C. C. MacDuffee of the University of Wisconsin and Professor C. G. Latimer of the University of Kentucky alternated as chairmen of the Algebra Section, and Professor U. G. Mitchell of the University of Kansas presided at the Section on Analysis and Geometry. The Symposium Lecture and the General Session on Saturday were presided over by Dean E. B. Stouffer of the University of Kansas and Dean W. H. Bussey of the University of Minnesota, respectively.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by *t* were read by title. Papers numbered 1 to 8 were given in the Algebra Section, those numbered 9 to 18 in the Analysis and Geometry Section, and those numbered 19 to 46 in the General Session. Paper number 5 was read by Professor G. Y. Rainich.

1. *Rational equivalence of a form to a sum of pth powers* (preliminary report), by Dr. Rufus Oldenburger. (Abstract No. 42-7-285.)

2. *On the rational equivalence of pairs of Hermitian matrices*, by Mr. A. E. May. (Abstract No. 42-5-159.)

3. *On ideals in a quaternion ring and the representation of integers by Hermitian forms*, by Professor C. G. Latimer. (Abstract No. 42-5-151.)

4. *On a fundamental theorem in matric theory*, by Professor C. C. MacDuffee. (Abstract No. 42-5-170.)

5. *Note on group postulates*, by Mr. Herbert Boggs and Professor G. Y. Rainich. (Abstract No. 42-5-168.)

6. *On quadratic rings and the arithmetics of quaternion algebras*, by Dr. Ralph Hull. (Abstract No. 42-5-155.)

7. *Canonical form as a many-valued function in matrix space* (preliminary report), by Mr. R. W. Wagner. (Abstract No. 42-5-167.)

8. *A simple recursion method for solving a set of linear equations*, by Professor Cornelius Lanczos. (Abstract No. 42-5-173.)

9. *Arithmetic genus of an algebraic variety*, by Dr. S. F. Barber. (Abstract No. 42-3-131.)

10. *Differential geometry of a surface at a planar point*, by Professor V. G. Grove. (Abstract No. 42-3-117.)

11. *A generalization of the elementary transcendental functions by means of a theorem of Volterra*, by Professor H. P. Thielman. (Abstract No. 42-3-128.)

12. *Identities between certain sums and integrals*, by Professor Fritz John. (Abstract No. 42-5-171.)

13. *New basic necessary conditions and methods in the calculus of variations*, by Professor W. S. Kimball. (Abstract No. 42-5-153.)

14. *The interrelations of the fundamental solutions of the hypergeometric equation*, by Mr. L. E. Mehlenbacher. (Abstract No. 42-5-169.)

15. *A note on critical points*, by Professor Morris Marden. (Abstract No. 42-5-234.)

16. *An analog to the Fekete theorem for the critical points of Green's function*, by Professor Morris Marden. (Abstract No. 42-3-124-*t*.)

17. *On the relation of Appell polynomials to orthogonal polynomials*, by Dr. M. S. Webster. (Abstract No. 42-3-127-*t*.)

18. *On the topology of a two-parameter, non-metric, and non-separable space*, by Dr. E. G. Harrell. (Abstract No. 42-5-165-*t*.)

19. *On the expansion of a function holomorphic in distinct regions*, by Dr. P. W. Ketchum. (Abstract No. 42-3-118.)

20. *A lemma on the topological index*, by Professor Tibor Radó. (Abstract No. 42-3-125.)

21. *Definitely self-adjoint boundary-value problems*, by Professor G. A. Bliss. (Abstract No. 42-5-156.)

22. *A system of ordinary linear differential equations with two-point boundary conditions*, by Dr. W. T. Reid. (Abstract No. 42-5-157.)

23. *The diagonal summation of double Fourier series*, by Professor Dunham Jackson. (Abstract No. 42-5-158.)

24. *Functions of bounded variation in two or more variables*, by Professor R. L. Jeffery. (Abstract No. 42-5-160.)

25. *Proof of the ideal Waring theorem for exponents exceeding six*, by Professor L. E. Dickson. (Abstract No. 42-5-231.)

26. *Remark on the deviation from analyticity of a non-analytic function*, by Professor G. Y. Rainich. (Abstract No. 42-5-166.)

27. *Sub-$F(x; \alpha, \beta)$ functions*, by Dr. E. F. Beckenbach. (Abstract No. 42-5-163.)

28. *Remarks concerning mathematical controversies* (preliminary report), by Professor Henry Blumberg. (Abstract No. 42-5-152.)

29. *On transformations in the plane.* I and II, by Professor Tibor Radó. (Abstract No. 42-3-119-*t*.)

30. *Proof that every positive integer is a sum of four squares*, by Professor R. D. Carmichael. (Abstract No. 42-3-120-*t*.)

31. *On numbers of the form* $a^2+\alpha b^2$, by Professor R. D. Carmichael. (Abstract No. 42-3-121-*t*.)

32. *On non-homogeneous linear differential equations of infinite order with constant coefficients*, by Professor R. D. Carmichael. (Abstract No. 42-3-122-*t*.)

33. *On a classification of integral functions by means of certain invariant point properties*, by Professor R. D. Carmichael, Dr. W. T. Martin (National Research Fellow), and Dr. M. T. Bird. (Abstract No. 42-3-123-*t*.)

34. *A generalized element of decomposition for doubly periodic functions*, by Dr. G. D. Nichols. (Abstract No. 42-3-132-*t*.)

35. *A remark on the area of surfaces*, by Professor Tibor Radó. (Abstract No. 42-3-126-*t*.)

36. *Ranks of* 4*-way matrices*, by Dr. Rufus Oldenburger. (Abstract No. 42-3-129-*t*.)

37. *Arithmetic invariants of binary cubic and binary trilinear forms*, by Dr. Rufus Oldenburger. (Abstract No. 42-3-130-*t*.)

38. *Integration of certain simple* (*step*) *functions*, by Dr. H. P. Doole. (Abstract No. 42-5-154-*t*.)

39. *Asymptotic Waring theorems*, by Professor L. E. Dickson. (Abstract No. 42-5-232-*t*.)

40. *Generalization of the associative law*, by Dr. Max Zorn. (Abstract No. 42-5-161-*t*.)

41. *On the second fundamental theorem in the theory of alternative algebras*, by Dr. Max Zorn. (Abstract No. 42-5-162-*t*.)

42. *A certain mean value problem in statistics*, by Professor A. T. Craig. (Abstract No. 42-5-164-*t*.)

43. *The integers represented by sets of binary biquadratic forms* (preliminary report), by Dr. O. K. Sagen. (Abstract No. 42-5-172-*t*.)

44. *Concerning linear difference equations containing a parameter*, by Professor W. J. Trjitzinsky. (Abstract No. 42-5-233-*t*.)

45. *Stresses in moderately thick rectangular plates*, by Mr. Archie Higdon and Professor D. L. Holl. (Abstract No. 42-5-235-*t*.)

MARK H. INGRAHAM,
*Associate Secretary*

## THE SPRING MEETING IN BERKELEY

The three hundred thirty-second meeting of the American Mathematical Society was held at the University of California on Saturday, April 11, 1936, Professor H. F. Blichfeldt presiding.

The attendance included the following twenty-seven members of the Society:

H. C. Ayres, H. M. Bacon, E. T. Bell, B. A. Bernstein, H. F. Blichfeldt, F. A. Butter, G. C. Evans, A. L. Foster, E. C. Goldsworthy, O. G. Harrold, Frank Irwin, R. D. James, D. N. Lehmer, S. H. Levy, J. C. C. McKinsey, C. B. Morrey, F. R. Morris, H. P. Robertson, Alvin Sugar, L. H. Swinford, J. M. Thompson, V. H. Tingey, J. V. Uspensky, Morgan Ward, A. R. Williams, B. C. Wong, H. S. Zuckerman.

Luncheon for members and their guests was served at the Faculty Club.

Titles of papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title. Mr. Webb was introduced by Professor Bell; Messrs. Highberg, Paxson, Mersman, Taylor, and Elconin by Professor Michal.

1. *Postulates for abelian groups and fields in terms of non-associative operations*, by Professor B. A. Bernstein. (Abstract No. 42-5-133.)

2. *On Boolean functions of n variables*, by Mr. J. C. C. McKinsey. (Abstract No. 42-5-134.)

3. *Sets on which a harmonic function becomes positively infinite*, by Professor G. C. Evans. (Abstract No. 42-5-135.)

4. *Pseudo-polynomials in abstract spaces*, by Mr. I. E. Highberg. (Abstract No. 42-5-136.)

5. *The differential in abstract linear spaces with a topology*. II, by Professor A. D. Michal and Mr. E. W. Paxson. (Abstract No. 42-5-137.)

6. *Note on formulas for the number of representations of an integer as a sum of 2h squares*, by Dr. R. D. James. (Abstract No. 42-5-138.)

7. *A new summation method for divergent series*, by Mr. W. A. Mersman. (Abstract No. 42-5-139.)

8. *Note on an inversion formula*, by Professor E. T. Bell. (Abstract No. 42-5-140.)

9. *On the solutions of quasi-linear elliptic partial differential equations*, by Professor C. B. Morrey, Jr. (Abstract No. 42-5-141.)

10. *The elementary analysis of the calculus of sequences*, by Professor Morgan Ward. (Abstract No. 42-5-142.)

11. *Development of the algebra of n-valued logic*, by Mr. D. L. Webb. (Abstract No. 42-5-143.)

12. *On a problem by John III Bernoulli*, by Professor J. V. Uspensky. (Abstract No. 42-5-144-*t*).

13. *Existence theorem on implicit vector functions*, by Dr. I. M. Hostetter. (Abstract No. 42-5-145-*t*.)

14. *Abstract euclidean spaces.* IV, by Professor A. D. Michal, Mr. I. E. Highberg, and Mr. A. E. Taylor. (Abstract No. 42-5-146-*t*.)

15. *Analytic functions in general analysis.* II, by Mr. A. E. Taylor. (Abstract No. 42-5-147-*t*.)

16. *An existence theorem for divergent series*, by Mr. W. A. Mersman. (Abstract No. 42-5-148-*t*.)

17. *Differential properties of abstract transformation groups with abstract parameters.* II, by Professor A. D. Michal and Mr. Victor Elconin. (Abstract No. 42-5-149-*t*.)

18. *General tensor analysis*, by Professor A. D. Michal. (Abstract No. 42-5-150-*t*.)

In the absence of Associate Secretary T. M. Putnam, Professor E. C. Goldsworthy acted as secretary.

T. M. PUTNAM,
*Associate Secretary*

## PELSENEER ON MATHEMATICAL THOUGHT

*Esquisse du Progrès de la Pensée Mathématique. Des Primitifs au IX^e Congrès des Mathématiciens.* By J. Pelseneer. Paris, Hermann, 1935. 162 pp.

Anyone seeing this little volume in the window of a bookshop in Paris would be apt to look upon it as simply another popular survey of the history of elementary mathematics. Such surveys of various fields are numerous, particularly in France and Germany, and they serve a good purpose. A second glance at the title, however, should tempt the inquisitive observer to open the first few pages. If this be done, he will probably find much that is new to him and will see the beginning of a treatment of the subject not generally found in the popular books of the day.

M. Pelseneer begins by calling attention to Boutroux's *L'idéal scientifique des mathématiciens dans l'antiquité et dans les temps modernes* (Paris, 1920). This work sought to reveal the hidden spirit which, at different times, inspired and animated the researches of some of the world's great mathematicians. It is such a plan of approach to a new type of history, abandoning that of a series of statements in chronological order, that M. Pelseneer suggests and, to a limited degree, attempts. He seeks first the spirit of the times—or *"l'air du temps"*—in which some of the great discoveries were made and by which the leaders were influenced. Since, however, it is impossible in such a limited "esquisse" as this to set forth in detail all the influences that make for the discovery of even a single feature in the advance of mathematics, he aids the reader by references to a few recent publications from which material for further study can be drawn. These references are confined almost exclusively to French, German, and English sources, and all the quotations of modern material are in the first of these languages as being the most familiar to the author's readers. One remark which he makes in this connection is particularly interesting, namely, that mathematicians are the least tiresome of all conversationalists since they never talk about their own subjects; they may mention their achievements, but *"leur oeuvre s'accomplit dans le silence."*

The following list of chapters will serve to show the general scope of the work: I. The primitives, with subtitles (1) Number (the logical aspect), (2) Number (the mystical aspect), (3) The absence of geometry; II. The pre-Greek period, including (1) The Egyptians, (2) The Sumerians and Babylonians; III. The Greeks; IV. The Cartesian period; V. The 19th and 20th centuries.

The closing chapters relate to modern times, and therefore offer relatively little that is new with respect to material. This allows more space for the discussion of "la pensée mathématique" in the sense of the term as set forth in the early part of the book. For the mathematician this is naturally the most valuable feature of the work. It gives the author an opportunity of summarizing briefly some of the views of such scholars as Boutroux, Couturat, Darboux, B. Russell, Wavre, Poincaré, and the original works of several modern mathematicians.

The work lives up to its title so far as being a sketch, and also as relating largely to the primitives. Only a small part of the world is considered, however,

no reference being made to the Far East or to various lands including or connected with the European centers. Relatively few of the names and achievements of the world's most prominent mathematicians are mentioned, although many more would naturally be expected in any consideration of the *Progrès de la Pensée Mathématique*. It is to be hoped that M. Pelseneer will carry out on a large scale a further and more complete study of the hidden spirit which has inspired the greatest scholars. For example, what was the spirit that inspired Menaechmus, Deinostratus, Diocles, Heron of Alexandria, and other Greeks whose names are not mentioned? As a single problem, who will undertake to consider the influences which surrounded Leonardo Fibonacci? If a man like Dr. Solomon Gandz, well known for his valuable contributions to *Isis* and other publications, and with the necessary equipment in the Semitic, Latin and Greek, and modern European languages could find the time to make this study for Fibonacci alone—and I know of no other similar field which promises such interesting results—the work would be of great value and interest.

In the later period there are also such leaders as Napier, Desargues, Barrow, Cotes, Jacobi, Plücker, and Bolyai—to name but a few—each offering a problem of historical interest almost equal to that of Fibonacci, but generally with less of what we may call mathematical romance. Will M. Pelseneer have the time to extend his studies so as to include these and others of equal standing?

The book has a good index of names, some of them of less importance than those mentioned above; but for an index of topics the reader must resort to the less satisfactory *Table des Matières*.

DAVID EUGENE SMITH

## MONTEL ON UNIVALENT FUNCTIONS

*Leçons sur les Fonctions Univalentes ou Multivalentes*. By Paul Montel. Collection de Monographies sur la Théorie des Fonctions, publiée sous la direction de M. Émile Borel. Paris, Gauthiers-Villars, 1933. iv+159 pp.

This book reproduces, with some modifications, the author's course of lectures delivered at the Sorbonne during the winter of 1929. It contains an exposition of a multitude of interesting results of the theory of univalent functions and of their various extensions. This theory, the origin of which is in the general problem of conformal mapping of simply-connected domains, was given a powerful impetus by the "Verzerrungssatz" of Koebe and Bieberbach. Since then it has attracted the attention and efforts of many mathematicians and at present is perhaps one of the most developed branches of the modern theory of functions of a complex variable. The author starts with classical results, but gives a short account also of the most recent ones, of which, however, he often reproduces only statements, omitting proofs altogether. A consistent use of the theory of normal families which was introduced and developed with great success by the author contributes considerably to the general elegance and unity of exposition.

A brief description of contents follows. Chapter 1 (*Univalence et multivalence des fonctions analytiques*) contains the fundamental notion of the theory of univalent and multivalent functions together with simple facts concerning convex and star-shaped domains. Chapter 2 (*Ordre de multivalence des poly-*

*nômes*) treats of properties of univalent polynomials. Theorems concerning the roots of a pair of apolar polynomials (Grace-Szegö) play an important role here. Chapter 3 (*Familles de fonctions univalentes ou multivalentes*) introduces the notion of normal and quasinormal families with various applications, including a solution of the general problem of conformal mapping of simply-connected domains. Chapter 4 (*Familles particulières de fonctions univalentes*) deals with various applications of the "Verzerrungssatz." A simple solution (due to R. de Possel) of the problem of mapping a multiply-connected domain into a plane with parallel slits should be mentioned here separately. Chapter 5 (*Domaines d'univalence ou de multivalence de certaines fonctions*) is devoted mainly to a discussion of domains of univalency of entire functions or of functions bounded in the unit circle. An interesting analogy is pointed out here between problems of this type and problems concerning exceptional values of functions (Picard theorem). Chapter 6 (*Propriétés extrémales des fonctions univalentes*) treats of some simple extremal properties of the mapping function and of its approximation by means of Tchebichev polynomials. Chapter 7 (*Domaines couverts par les valeurs des fonctions régulières*) gives an exposition of results connected with Bloch's theorem. The book ends with a note by M. Cartan on the theory of univalent functions of two complex variables.

The usefulness of the book is somewhat impaired by the presence of numerous misprints and lapses. Thus, after being given the usual definitions of a convex domain $D$ (or a domain star-shaped relative to a point $O$) the reader is told that as a point $A$ describes the contour of $D$, the tangent at $A$ turns through an angle of $2\pi$ (or the vector $OA$ turns through $2\pi$), which rules out such familiar domains as a half-plane (or a plane with a slit along a segment of the axis of reals, extending to infinity). The reader is also told (p. 15) that if there exists a number $\phi$ such that $Re^{i\phi}zf'/f>0$, in $|z|<r$, and if $f(z)$ is analytic in $|z|<r$ and $f(0)=0$, then $f(z)$ is univalent in $|z|<r$ ($f(z)=z^2$?). In another place (p. 10) the reader is told that if a function $f(z)$ is analytic in a closed domain $(d)$ where $f'(z)\neq 0$, and if there is no interior point in $(d)$ at which $f(z)$ assumes the same value as on the boundary, then $f(z)$ is univalent in $(d)$ ($f(z)=z^2$, $\frac{1}{2}\leqq|z|\leqq 1$?). Finally, the footnote 2 on page 5 calls for the following observation. The author refers to his note in the Comptes Rendus (vol. 156 (1913), pp. 1820–1822) concerning an important extension of Cauchy's Theorem, to the effect that if $Z=f(z)=X+iY$ is continuous in a domain $(d)$ and if the partial derivatives $\partial X/\partial x$, $\partial X/\partial y$, $\partial Y/\partial x$, $\partial Y/\partial y$ exist everywhere in $(d)$ and satisfy Cauchy-Riemann relations almost everywhere in $(d)$, then $f(z)$ is analytic in $(d)$. He states that this result was rediscovered later by Looman (Göttinger Nachrichten, 1923, pp. 97–108). The theorem in question was stated in the author's note above even in a more general form, namely, a necessary and sufficient condition that $pdx+qdy$ ($p$, $q$ continuous, $\partial p/\partial y$, $\partial q/\partial x$ exist everywhere) be a total differential is that $\partial p/\partial y=\partial q/\partial x$ almost everywhere. However no proof of this more general theorem has ever been published. Moreover, Looman's proof of the Cauchy theorem is incorrect and the only correct proof of the Cauchy theorem under the general assumptions above, existing in the literature, is due to Menchoff (see S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 243).

J. D. TAMARKIN

## MOORE ON GENERAL ANALYSIS—I

*General Analysis.* Part I. *The Algebra of Matrices.* By Eliakim Hastings Moore with the cooperation of Raymond Walter Barnard. Memoirs of the American Philosophical Society, Philadelphia. Vol. I, 1935.

Eliakim Hastings Moore will always be counted among the heroic figures of American mathematics. His life was devoted to research and to the implanting of respect for research in the minds of others. As head of the department at the University of Chicago he came into contact at one time or another with a large proportion of the present generation of mathematicians in America, and many there are who derived from him an ambition to contribute something to the science. Professor Moore's personality was dominating. He was unusually careful in matters of rigor, very severe with students who were guilty of loose thinking, but quick to forgive and encourage.

Like Weierstrass and Lie, Moore published very little of his later work. This was partly because of the comprehensive nature of his problem, and the interdependence of its parts. It was also due to the extremely high standard which Moore set for himself, and his severe self-criticism. He was continually changing and polishing his work and was never willing to give it his final approval. But now his work is finished, and with the sympathetic and able assistance of Professor Barnard it is given to the world.

Moore's *General Analysis* will be published in four volumes of which this, *The Algebra of Matrices*, is the first. The other volumes will be *The Fundamental Notions of General Analysis, Generalized Fourier Series and Modular Spaces*, and *The Characteristic Value Problem in General Analysis.* This first volume contains a preface by G. A. Bliss and an introduction to the entire series by Barnard. In this introduction is a sketch of Moore's first General Analysis theory, and of the historical development of the second (present) theory. The reader is then taken into the author's confidence, the underlying ideas and motives of the General Analysis are explained, its triumphs and temporary failures analyzed, and the changes noted which had to be made to mold it into its final form.

One of the striking features of Moore's work is his extensive use of symbolism. His notation is founded upon that of Peano but it has been greatly modified and extended. To a person not familiar with it, it appears formidable, but actually the system is so consistent that it becomes clear after only a short study. To Moore it was not a system of shorthand, but a medium in which ideas could be expressed with greater clarity and rigor. By its use he was led to finer distinctions than most mathematicians are accustomed to make, and to a better realization of his underlying assumptions. Then, too, the careful statement of a theorem in symbolism often gave him a clue to a possible method of proof.

"*The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.*" This quotation from Moore's New

Haven Colloquium Lectures (1906) is a declaration of faith which inspired the whole development of his General Analysis. Thus in 1915 he noted that his earlier theory interpreted for the $n$-dimensional Euclidean manifold appeared to rest upon the presence of a finite properly positive (positive definite) Hermitian matrix. He also noted that the theory of quadratic forms in infinitely many variables rested similarly on the infinite identity matrix. Finally, the work of Hellinger could be shown to depend on a symmetric positive definite matrix of infinite order. In attempting to distill from these diverse situations the essence which they have in common, the second General Analysis theory was formulated, resting upon an algebraic foundation of Hermitian matrices.

This first volume is divided into but three chapters, the first of which is concerned with matrices with elements in a non-commutative field, or quasi-field, or number system of type $A$, as it is called. The quasi-fields of most importance in the theory are the rational, real, and complex fields and their subfields, and the quasi-field of real quaternions. Functions on a single general range $\mathfrak{P}$ to a second range (usually a number system) are called vectors. Functions on two ranges to a third range are called matrices—thus, dropping Moore's terminology for a moment, a matrix $A=(a_{ij})$ is merely a function of the two variables $i$ and $j$. Since the composite of the two ranges may be regarded as a single range, every matrix may be regarded as a vector. Conversely, every vector may be regarded as a matrix one of whose ranges is singular. The (open) product of two vectors $\xi$, $\eta$ is a matrix whose trace $S\xi\eta$ is their inner product. If $\kappa$ is a matrix and $\xi$ a vector, then $S\kappa\xi$ is a vector whose components are the inner products of the rows of $\kappa$ with $\xi$. Finally, the operation $S$ operating on two matrices gives their ordinary matric product.

No satisfactory definition of determinant being at hand, the usual development of matric algebra is not possible and an alternative approach is made necessary. The matrix $\delta$ is the identity for the composition process, and matrices $\lambda_r$, $\lambda_l$ such that $S\kappa\lambda_r=\delta$, $S\lambda_l\kappa=\delta$, are called right and left reciprocals of $\kappa$, respectively. A finite matrix is non-singular if it has both types of reciprocal, in which case each is unique and the two are equal. A minor is defined as usual, and the rank of a finite matrix is defined as the order of a non-singular minor of maximum order, the rank of the zero matrix being zero. Thus the important theorems on rank are obtained without using the concept of determinant.

For a quasi-field there are two types of linear dependence, left and right. The rank of a matrix gives the number of columns in every maximal set of right linearly independent columns, or left linearly independent rows. There are two types of systems of linear equations, and for each there is an exact analog of the ordinary theory. Thus the general right-hand solution of a non-homogeneous system can be expressed as the sum of any particular solution and the general solution of the corresponding homogeneous system. The latter is a right linear combination of any $n-r$ right linearly independent solutions, $r$ being the rank of the matrix and $n$ the number of unknowns.

Every matrix is shown to be a product of elementary matrices, or generators of the matric group, as they are termed. Two matrices $\kappa_1$, $\kappa_2$ are called equivalent if $\kappa_2$ (say) can be factored in the sense of composition into a product of three matrices, the middle one being $\kappa_1$ and the others non-singular. A matrix of rank $r$ is in normal form if it has zeros everywhere except on the main diagonal,

and if this contains $r$ 1's and the other elements are 0. Every matrix can be reduced to normal form by multiplying it on either hand by elementary matrices.

In the second chapter it is further assumed that the quasi-field admits a conjugate process or reciprocal automorphism of period two. That is, corresponding to every number $a$ there is a conjugate $\bar{a}$, whose conjugate is again $a$, such that the conjugate of $a_1+a_2$ is $\bar{a}_1+\bar{a}_2$ and the conjugate of $a_1a_2$ is $\bar{a}_2\bar{a}_1$. For a (commutative) field the correspondence of each number with itself satisfies these conditions. For complex numbers and quaternions the ordinary conjugate process satisfies. Relative to this relation of conjugacy, an Hermitian matrix is defined as a matrix whose transpose is obtained by replacing each element by its conjugate. For Hermitian matrices a definition of determinant is possible. This definition is a little long to be given here with exactness, but it may roughly be illustrated for $n=4$ as follows: Let (234) (1) be a partition of the integers 1, 2, 3, 4. To this partition let correspond the product $(a_{23}a_{24}a_{42}+a_{24}a_{43}a_{32})\,(a_{11})$ with a proper sign attached, the subscripts corresponding to all permutations of 234 starting with 2. For $n=7$, to the partition (123) (4567) would correspond a product of two parentheses, the first inclosing two terms, the second six. The determinant $|a_{ij}|$ is the sum of the products corresponding to all partitions. Since each group of terms in parentheses is scalar, the parentheses are commutative with each other, and the determinant is a scalar. If the elements belong to a field, the ordinary determinant is obtained.

The Laplace formulas for the development of a determinant by elements of a row or column are valid. In developing by the elements of a row the cofactors occur on the right, while for the development by the elements of a column, the cofactors appear on the left. The usual criteria hold for determining the rank of an Hermitian matrix from the determinants of the principal minors. Every Hermitian matrix satisfies its characteristic equation. An Hermitian matrix has an adjoint which is Hermitian. Two matrices $\kappa_1$ and $\kappa_2$ are conjugate if there exists a non-singular matrix $\phi$ such that $\kappa_2=SS\phi^*\kappa_1\phi$, where $\phi^*$ is the conjugate-transpose of $\phi$. Under such transformations every Hermitian matrix can be reduced to diagonal form. Next Hermitian bilinear functional operators are introduced, and the chapter closes with a consideration of the subspaces of vectors of a given space and the orthogonal complements of these spaces, orthogonality being relative to an Hermitian operator.

In the third and last chapter, an order relation is introduced into the quasi-field with conjugate relation. To this end it is postulated that among the scalars of the quasi-field $\mathfrak{A}$ occur the norms $\bar{a}a$ of all the numbers $a$ of $\mathfrak{A}$, and these norms have the order relations of the positive numbers. It is now possible to define positive and properly positive Hermitian matrices and forms. An Hermitian matrix is positive if each of its finite principal minors has determinant $\geqq 0$, properly positive if each is $>0$. The Hermitian form $F_{\bar{\mu}\mu}$ is positive if it is $\geqq 0$ for every vector $\mu$, properly positive if, further, it is 0 only when $\mu=0$. The fundamental result is obtained that an Hermitian bilinear form is (properly) positive if and only if its corresponding Hermitian matrix is (properly) positive. An analog of Sylvester's law of inertia is proved.

At this point there is introduced the notion of general reciprocal for any finite matrix $\kappa$. It may briefly be defined as the matrix $\lambda$ such that $SS\lambda\kappa=\kappa$, uniqueness being secured by demanding that each column of $\lambda$ belong to the

linear space determined by the columns of $\kappa^*$, and the conjugate of each row of $\lambda$ belong to the linear space of the columns of $\kappa$. This general reciprocal reduces to the ordinary reciprocal if $\kappa$ is non-singular. The general reciprocal of a (positive) Hermitian matrix is again (positive) Hermitian. It is not always true, however, that the reciprocal of a product is the product of the reciprocals in reverse order.

The book closes with a discussion of the Gramian matrix arising from a positive Hermitian bilinear equation, and the problem of orthogonalization with respect to such an operator. Every finite or infinite sequence of vectors can be orthogonalized with respect to the positive Hermitian bilinear operator $B$, and if $B$ is proper, the orthogonalization is unique.

Professor Barnard and Dr. Coral are to be congratulated upon the excellent presentation. Besides the general introduction, each chapter is preceded by an adequate summary and explanation of its contents. While Moore's notation is preserved throughout, the principal definitions and theorems are explained in words as well as in symbols, and everything possible has been done to make the book comprehensible to the reader. The difficult typography is excellently done.

Finally, it must not be forgotten that Barnard has himself made important contributions to the work which are more than editorial in character, although modesty has forbidden him to point these out.

C. C. MacDuffee

## REICHENBACH ON PROBABILITY

*Wahrscheinlichkeitslehre.* By Hans Reichenbach. Leiden, Sijthoff, 1935. 451 pp.

A complete and perfect work on probability would tell what probability is, would give all mathematical developments in full, and would act as a guide to the application of the theory. The present work by Reichenbach has much of mathematical interest, and deals with a wide range of uses of the probability concept; but its greatest importance comes from the fact that it brings us definitely closer to an understanding of probability itself. This achievement is the more valuable in view of the many conflicting definitions which have been given. Reichenbach himself uses three definitions. The first is purely axiomatic; the second and third, frequency-limit and point-set definitions, are proved to be applications of the first. Other concepts, such as that based on equal probability or the psychological ones, he shows to be either untenable or reducible to the frequency-limit definition.

The author comes to the study of probability from the fields of both physics and philosophy. As a physicist he sees that we can not know that any rigorous laws rule the universe; nor yet—assuming this to be a world of law—the exact formulation of any law. Nor—assuming we knew a precise law to hold—can we measure with precision either the present state on which that law would base a prediction, or the future state which would verify it. So it is that a physicist, even though he ignore the added uncertainties introduced by quantum mechanics, finds the theory of probability essential to an understanding of our relations with nature. "We are dealing here, not alone with problems of

mathematics and mathematical physics,—we are dealing with our ultimate knowledge of nature altogether, with the question of what our statements concerning reality actually mean." As a philosopher and logician, Reichenbach comes to his study of probability confident that a formalization of the theory is the best means for reaching clarity. Hence he begins with a brief but adequate account of formal logic and of the significance of a system of axioms—an account derived largely from Russell and Whitehead, Hilbert, and Ackermann.

The following chapter (*Elementary calculus of probabilities*) defines probability as a number, a function of two classes of events. The calculus is built up on axioms which, at many points, generalize the logical axioms of strict implication. Thus, in place of "the inclusion of event $x_i$ in class $O$ implies the inclusion of event $y_i$ in class $P$ with certainty," we have "$x_i \epsilon O$ implies $y_i \epsilon P$ with probability $p$." In abbreviated symbolism $W(O, P) = p$. The definition and the symbol both emphasize the fact that probability depends not only on the conclusion but also on the premise; the probability that a die fall with the 6 uppermost depends on the set of throws used as a basis. Thus, consider the set of throws in which 6 is uppermost when the die is two inches above the table.

The axiomatic, abstract definition of probability does not assign to a particular pair of sequences a definite number—except that 1 and 0 are the values of the probability when $O$ implies $P$ and not-$P$ respectively. Neither does it afford us certainty that, for a given pair, the probability exists, nor even—in general—does it offer any criterion as to its existence. In fact, it is conceivable that various concrete definitions, each uniquely fixing the value of probability whenever it applied, would apply to different classes of sequence pairs.

The axioms were, naturally, devised in order to fit a particular concrete concept. They are satisfied if we define $W(O, P)$ as the limit of the ratio of the number of cases where $O$ and $P$ both hold to the total number of cases of $O$; and this is the only content given to the axioms so far as discrete sequences are concerned. Among the topics particularly well treated in the elementary discussion are Bayes' rule and the independence of events.

Many applications of the theory of probability depend, not only on the existence of probability for the given sequence, but also on the existence and value of that number for certain sub-sequences. Von Mises, in fact, goes so far as to deny the name of probability to sequences unless they satisfy a "principle of irregularity" which makes very sweeping demands of this nature. Reichenbach has analyzed this concept and found it untenable. He does, however, require of all infinite sequences yielding a probability that that number be unchanged by the omission of any finite number of terms at the beginning. He defines and uses frequently "normal sequences"—those in which a sub-sequence has the probability of the given sequence if inclusion of a term in the sub-sequence is determined either by the nature of other terms at fixed distances before it, or by its place in a regular rotation. Important abnormal sequences exist. Thus the positions of a gas molecule in an enclosed space at very short intervals will show after-effects. A brief study of such hereditary phenomena and of lattices, sequences of sequences, concludes the chapter.

In the short but adequate chapter on probability sequences in which the marks of the possible results of a trial are numbers—on mean values and

variance—we may question the use of "Mittelwert" for the average of a finite number of quantities, "Durchschnitt" for that of an infinite sequence.

Reichenbach has given much attention to "geometric" probability—in which an additive measure of point sets may replace frequency and give a second concretion of the axioms. The chapter on this topic is valuable, both for its clear exposition of the concepts and for its variety of applications. This is a chapter, however, in which one could wish more care had been used. On page 225 there is mention of the "most probable value" in a continuum—since, in general, each value will have probability zero, the phrase, common as it is, should be avoided. Reichenbach, comparing surfaces of rotation with "Streckungsflächen" (p. 230)—those of type $z=\phi_1(x)\ \phi_2(y)$—names the latter a more general type, implying to the unwary reader that they include the former. The reviewer would prefer that the book did not refer to "infinite summation of infinitesimal terms" (p. 239). On page 248 Reichenbach divides a time-interval $\tau$ into a finite number of sub-intervals; stating that the number of changes from $P$ to not-$P$ in $\tau$ equals the number of sub-intervals with $P$ at the beginning and not-$P$ at the end, he quite ignores the possibility of two or more reversals within a sub-interval.

Chapter 7, on *Frequency properties of probability sequences*, contains results notable for the combination of the axiomatic definition of probability and of the frequency concept. Thus Bernoulli's theorem assumes the following form: Given a sequence in which the *probability* of a certain event is $p$ and an arbitrary $\delta$, then the probability that the *relative frequency* of the event lie between $(p-\delta)$ and $(p+\delta)$ approaches 1 as $n$ becomes infinite. Thus, as an undefined probability occurs in the premise, another undefined probability replaces it in the conclusion—on a higher level, as it were. It should be noted that this is meaningless unless the sequence is immersed in a class of sequences.

For normal sequences Pólya has strengthened this theorem. According to him, 1 is the limit, not only of the probability that the $n$th relative frequency be in the interval named, but also of the probability that it remain there for all succeeding values of $n$. This is almost enough to identify the abstract probability with the limit of relative frequency. A further axiom is all that is needed in order to make this identification. The axiom is the plausible one: "If an event $C$ is to be expected in a sequence with a probability converging to 1, it occurs in the sequence at least once." For normal sequences, then, which by definition have probabilities, the limit of the relative frequency must be the probability.

After a chapter on probabilities of higher type (probabilities of probabilities, dispersion and the like) come the concluding chapters, philosophical in nature, with valuable contributions to the question of the applicability of the theory to the world of experience. The chapters are entitled *The problem of application* and *The logic of probability*. "A scientific philosophy must require that every statement be verifiable." Have we a right, then, to say that the probability of an event is 2/3? If we accept the frequency interpretation, we can not know, from the finite number of observations we can make, whether a limit is being approached; or, if so, whether that limit is 2/3. We must be consoled—if it is consolation—by realizing that no measurement whatever can do more than this for us. Assuming, however, that we could know the prob-

ability, how would that knowledge guide our action? For we act in a single case, on which probability has nothing to say. Reichenbach answers:—with a bet or an assumption. We act, he says, as if the outcome of highest probability were a certainty. This is, of course, too simple; if an aeroplane has 9/10 probability of escaping disaster, many will nevertheless decline to be passengers. As for gaining an estimate of the value of the probability itself, Reichenbach sets up a rule—a decidedly weak rule—according to which we assume the probability to be near to the frequency already obtained. *If* a probability exists *then* this procedure will, in time, yield it within any required degree of accuracy—no one can say in how much time.

Moreover, the limit of frequency may not exist. If, for example, a "constant" in a natural law were to oscillate between two extremes in ever lengthening periods, the relative frequency of a phenomenon dependent on it might have no limit. Reichenbach mentions sequences similar to these—sequences, however, in which the transition between one probability and another is abrupt. These "comprehensible" (übersehbare) sequences are not sharply defined; yet our author holds that, if the sequences of nature are of this type, we can eventually prophesy future events with as much justification as if we had probability-sequences. Whether this is true can not be guessed until we have a careful definition; and once we had it, the thorough discussion of comprehensible sequences would require a treatise of length comparable to that of this book itself.

The conclusion as to the applicability of the theory of probability is this: We do not and cannot know whether we live in a world of sufficiently permanent lawfulness to justify prophesying the future on the basis of the past. Yet, past experience is the only guide to the future which we have; it may fail us, but reliance on it still remains our "best bet."

The book is not furnished with exercises for training the reader, but it is replete with illustrations from many fields of experience. Thus we have examples from physics, medicine, factory production, games of chance, astronomy, transportation, and so on. In some instances the notation should be clearer. Thus, on pages 260 and 272 the subscript $m$ in $H^*_m$ denotes the number of successes and $H$ itself is a proposition, whereas, on page 273, $H_n(O, p^{k_i})_i$ is a number, and the subscript the number of trials. In the last sentence of page 380, $W(y_1 \text{,} x_1)$ is the probability *with which* $y_1$ implies $x_1$, but $W(y_1)$ is the probability *that* $y_1$ is true.

By the omission of a few items (proofs concerning the Gaussian distribution, Stirling's formula, and Bernoulli's theorem, as well as tables of the probability integral) the author fails to make the book mathematically self-contained. This is the more to be regretted, since completeness in these respects would have required so little additional space.

Reichenbach's book inevitably leaves many questions on the nature of probability still unanswered. But it marks such distinct advances in our knowledge concerning the concept that all mathematical and philosophical libraries should contain it; every thoughtful reader will be grateful for the new insight it brings to the subject.

E. S. ALLEN

## DIRAC ON QUANTUM MECHANICS

*The Principles of Quantum Mechanics.* By P. A. M. Dirac. Second edition. Oxford, Clarendon Press, 1935. xii+300 pp.

The first edition of Dirac's *Principles of Quantum Mechanics* appeared in 1930.* The second edition is essentially the same book, but is in many respects clarified and recast in exposition, with a shift in emphasis to the non-relativistic conception of *state*, and with additional matter, in particular, a chapter on the quantization of the electromagnetic field.

In the introductory generalities, comprising the preface and first chapter, such matters are discussed as the inadequacy of the classical theory with its determinism and causality, and the notions of states, probability, and the principles of superposition and indeterminacy. The non-technical reader is apt to be misled by the form of the discussion, which is that of a logical analysis of experimental data with the drawing of necessary physical and philosophical conclusions. Yet we believe that this is not at all what the author is undertaking. One hardly needs to be a practiced epistemologist to be aware of the order of magnitude of philosophical analysis required to treat adequately such questions as those of causality and indeterminacy, of which the author disposes—too often with scarcely more than the conventional phrases. Nor is unusual penetration needed to perceive that the experimental evidence claimed as overthrowing classical physics and establishing the new theory, requires a far more penetrating critique. As a pure matter of logic, one is left with the impression that all might about as well have been argued in the opposite direction. We believe that what the author has attempted to do here is a task far more important for the needs of a student of contemporary physics. He is attempting to *prepare the intuition* of the reader, so that the old theory becomes replaced by the new by a sort of *intuitive* necessity, and so that the form and substance of the new theory may be apprehended not only as a logical abstraction but in terms of a new conceptualism—almost a new imagery. In this respect he has made a distinguished contribution. The present edition is far clearer at this point than the first.

The next section of the book, from the second to the fifth chapters, deals with mathematical preliminaries and their physical interpretation: the linear vector spaces and operators and their representations, the transformation theory, and the quantum conditions. The point of view is to regard the aggregate of states of a system somehow as forming a linear vector space, the elements of which both are and are not the states (there is an indeterminate phase factor!), and the linear transformations of which are *en rapport* with the observables of the system. The familiar spaces of the quantum theory, com-

* Reviewed by me in this Bulletin, vol. 37 (1931), p. 495. The observations in the review of the first edition apply to a considerable extent to the second; we take the occasion of the present review to treat a rather different side of the question.

posed of number-sequences or of wave functions, together with their matric, differential, and integral operators, appear a posteriori as *representatives* of the states and observables, the agency in this representation being that embodiment of mathematical faith, the expansion theorem. A notation is introduced for the representatives, and is used consistently throughout the non-relativistic portions of the book; it has the great advantage of emphasizing the role of the analytical constructs as representatives, but the disadvantage of insufficient explicitness and practical inconvenience.

This view of the states and observables as prior realities, and the mathematical constructs as but their representatives, is one of the most notable features of the whole book. It is a conception in the grand manner; and with its aid everything is seen in a new clarity. Unfortunately it involves some grave difficulties. As a matter of conceptual experimental physics, there is a vagueness exceedingly difficult to dispel about such definitions as those of states and superposition, as general ideas; and the precise sense in which the states go to make up the linear spaces is scarcely more than indicated. And on the mathematical side, there is the question of the validity of the expansion theorem, and the related difficulty of the $\delta$-function. In the closed systems of the non-relativistic theory, to be sure, the states may be represented in Hilbert space: the expansion theorem applies, and the $\delta$-function may be dispensed with. But the quantum theory deals with non-closed systems, and also, with both closed and non-closed relativistic systems, in none of which is the expansion theorem in its general form valid. It is greatly to the credit of our author that he has not sought to fit these systems to the Procrustean bed of Hilbert space.

A possible exit from both orders of difficulty, as well as an explanation of the practical successes which the theory has achieved in spite of them, may lie in the observation that *the quantum theory is not a general theory*: it deals with a small number of exceedingly special physical systems, in which the states and observables are of a highly specialized character. This is not merely a fact of contemporary physics,—it is inherent in the very notion of a theory of primary phenomena. The postulational generality is purely linguistic—in obvious analogy with the form of the macroscopic theories of classical physics, dealing as they do with a high order of infinity of essentially distinct conceptual systems. And now in the few special cases of our theory which correspond to its only reality, the meaning in terms of experiment of state, observable, and so on, may be made quite definite and clear, and the expansion theorem actually is observed to apply. Yet the generality of form in which the theory is cast has a significance which becomes clear the instant that it is envisioned not as a system of propositions, but as a body of principles of the intuition, a pattern and guide in the unknown.

Along the lines of this clarification by specialization to actual systems, the $\delta$-function is quite simply treated: The reader will perceive—and at this point the author is far clearer than in the first edition—that the $\delta$-function is not a function at all, but a symbol for a category of definitions. In what concerns harmonic analysis, certain kernels involving a large integer are called for. And in the collision problems, all is made clear and rigorous once the treatment is paraphrased in terms of discontinuous additive set-functions. While such a transcription places quite a task upon the reader, we are of the opinion that

the present form of treatment has great value in showing the underlying unity behind the special mathematical situations into which the explicitly rigorous treatments tend to disintegrate.

The third part of the book, from the sixth to the tenth chapters, deals with the conventional non-relativistic theory: the equations of motion and their applications, including the hydrogen atom, angular momentum, and spin; perturbations and collisions; systems of similar particles. Here the author's conception of the mathematical objects as but representations of an underlying reality comes into its own. It receives an expression, for example, in the abstract algebra of observables, which forms so beautiful and characteristic a feature of the work. It is hardly a criticism to observe that this method could have been carried much further, and much that the author does with the aid of the representatives could have been done abstractly in a simpler manner, and one more in the spirit of the book.

The last three chapters deal with the relativistic equation of the electron and the electromagnetic field; they form the advanced portion of the work. Here the fundamental conceptions developed earlier have free play, and the result is a treatment which carries the ideas far deeper into the nature of the situation than do so many current discussions of the subject, where so much is buried under the analytical machinery. The relativistic conception of state is employed, with its wave-function defined throughout space-time. This is in sharp contrast with many of the treatments of the relativistic electron which, in order to confine the situation to Hilbert space, regard the wave-functions as representing a one-parameter family of states in 3-space, and are thus not relativistic in essence. It should be remarked that the author abandons his notation for the representatives, for no apparent reason, in the twelfth chapter. Our only criticisms are that we wish that the author had given a less abbreviated discussion of the positron, and had amplified that of the quantized electromagnetic field.

Aside from its function as a textbook on quantum theory,—a function which, as far as the uninitiated is concerned, it does not fulfill any too well—Dirac's *Quantum Mechanics* has a particular significance for the mathematician. Mathematics of the present day is resounding with the triumphs of the abstract method: we have abstract sets, abstract space, abstract algebra and arithmetic, abstract analysis, and so forth. And now here is a physicist who is approaching the highest department of modern physics with so completely abstract a point of view that he is regarded as a pure mathematician by many of his colleagues. Yet the general and abstract standpoint of our author is but the garb and vestment of a deeper thing, an inner principle, without which he has the vision to perceive that his abstract discipline would be but a lifeless form.

B. O. KOOPMAN

# SHORTER NOTICES

*Mathematical Tables.* Volume 5. *Factor Table.* Prepared independently by J. Peters, A. Lodge and E. J. Ternouth, E. Gifford. London, British Association for the Advancement of Science, 1935. xv+291 pp.

This volume, the fifth in a series prepared under the direction of the British Association for the Advancement of Science, contains a tabulation of the decompositions into prime factors of *all* integers between 0 and 100,000. In this respect, that it lists all such integers, the present table differs from most existing factor tables, since it has been the practice in the formation of the more extensive tables to omit multiples of 2 and 5, and often of 3 and 7, also.

Conforming to custom, this volume contains an introduction comprising a description of the various methods employed in the construction of the three independent manuscripts upon which the table is based. In addition, there are given several lists of errata found in other tables upon comparison with the present one, as well as an enumeration of the more important tables.

As to the table proper, the printing is excellent and the arrangement commendable on account of its simplicity with a consequent accessibility of data. There is appended to the body of the work a table of reciprocals, to eight significant figures, of primes between 10 and 10,000. These data are well adapted to the factorization of integers that does not exceed $10^8$, by means of calculating machines.

In conclusion, the reviewer believes that for accuracy and utility the table under discussion supercedes all others covering a similar range.

J. W. WRENCH, JR.

Bauschinger's *Tafeln zur Theoretischen Astronomie.* Second edition, revised by Gustav Stracke. Leipzig, Engelmann, 1934. v+192 pp.

The first edition of this collection of tables has been one of the stand-bys of astronomers all over the world ever since its appearance in 1901. Although primarily intended to be useful to computers of planetary and cometary orbits, all but a few special tables have found application in a much wider field.

During the years that separated the publication of the two editions the computational tools of astronomers have passed through a period of rapid evolution. Modern calculating machines are now in general use, and have replaced logarithmic methods wherever this simplifies or reduces the work. Convenient tables giving the natural values of trigonometric functions are now available, and the sexagesimal division of the degree is gradually giving way to the decimal division.

Dr. Stracke, who is the head of the division of minor planets of the Recheninstitut at Berlin, is well qualified to appreciate the value of these modern methods. He has succeeded in presenting a new edition that is thoroughly up to date, without neglecting the needs of those that are still using the older methods.

Few of the tables have been retained without some change. Without ex-

ception those changes are decided improvements, especially where the decimal division of the degree has been introduced. Among the new material are auxiliary tables for numerical integration, tables for conversion of ecliptical into equatorial elements and vice versa, for precession in rectangular coordinates, and for differential correction of orbital elements.

Of especial interest to non-astronomical users are the ten pages at the end of the volume giving a collection of formulas for interpolation, numerical differentiation and integration, tables of coefficients in interpolation formulas, a table of the probability integral, a page of mathematical and astronomical constants, and a list of logarithmic and trigonometric tables.

It is, of course, possible to suggest additional material that might have been included. Within its scope it is an excellent volume, the value of which is much enhanced by its beautiful printing.

DIRK BROUWER

*Die Mathematischen Hilfsmittel des Physikers.* By E. Madelung. 3d edition. Berlin, Springer, 1936. 13+381 pp.

The second edition of Madelung's *Hilfsmittel*, published in 1925, but still widely popular among working physicists, was rapidly losing its utility as the complexity of the mathematical tools employed by the physicist increased. The third edition represents a thorough rejuvenation; it enlarges the usefulness of the book and is likely to win for it many new friends.

The material of the book has been rearranged, and important additions have been made. Numerous items on which the reader previously had to seek elucidation in Courant-Hilbert or Whittaker and Watson may now be found adequately treated in Madelung's book. Even a short section on group theory has been included. The section on quantum mechanics, completely rewritten, contains a good summary of the principal facts, as well as sets of formulas (such as commutation rules between various operators) which are valuable to have at hand. The size of the book has grown from 283 to 381 pages.

HENRY MARGENAU

*Das Grundgesetz der Wellenfortpflanzung aus bewegter Quelle in bewegten Mittel.* By Karl Uller. Munich and Berlin, R. Oldenbourg, 1935. 138 pp.

This is a book devoted to the phenomena (mainly electro-magnetic) of wave motion in moving media. The writer does not accept the ideas of relativity and the spirit of the book may be inferred from the fact that in an appendix there are given no less than seventeen "Gegenbeweise" (whose dates of original publication extend over the interval 1910–1932) which claim to demonstrate the lack of validity of Einstein's theory. The present reviewer does not find these "proofs"—either taken singly or collectively—convincing. A single example of his inability to "go along" will have to suffice. On page 31 the author "proves" the commutativity of "local" and "substantial" time differentiations, but an examination of the "proof" makes it clear that the local time-rate of change of velocity is not taken into account.

F. D. MURNAGHAN

*Researches on Waring's Problem.* By L. E. Dickson. Carnegie Institution of Washington, Publication No. 464, July, 1935. v+257 pp.

Waring's problem is now entering its last phase: to find U such that *every* positive integer is a sum of $U=U(k)$ integral kth powers $\geqq 0$. As has long been known $U(3)=9$. But it is also known that *some* integers demand exactly 9 cubes. In this sense 9 is the true universal U for $k=3$. The Hardy-Littlewood analysis gives a number $N=N(k)$, depending only on k, such that every *sufficiently large* positive integer is a sum of at most N kth powers; for example, $N(4)=21$, whereas Waring conjectured $U(4)=19$.

To dispose finally of Waring's conjecture it remains to determine U(k), universal in the sense that $U(3)=9$ is, and to indicate how the integers which require fewer than U(k) kth powers can be calculated. This goal is in sight, owing to recent advances due to Winogradov and R. D. James (using in part the results of Winogradov). When all this is done the curious may wish to know the number of representations as a sum of a prescribed number of kth powers, $k>2$, in continuation of the classic results for $k=2$. Nothing of any consequence has been accomplished in this direction.

Dickson is led to consider $I=I(k)=2^k+g-2$, where g is the greatest integer $<(3/2)^k$. "No exception," he states, "has been found to the empirical result that every positive integer is the sum of I kth powers, so that I is the ideal value for a universal theorem." It is easily seen that $2^k g-1$ demands I kth powers. Reasons are given for stopping with U(17). For $k=7$ the Hardy-Littlewood N(k) is 192, while $I(k)=143$, and as k increases, $N(k)-I(k)$ increases rapidly, so that there is still plenty to shoot at: thus $N(13)=22639$, $I(13)=8384$. By an exhaustive analysis, Dickson finds $U(5)=54$, $U(7)=233$. For $k\geqq 11$ the new U(k) are less than the Hardy-Littlewood N(k). Thus after $k=10$ the new *universal* theorems are an improvement on the older *asymptotic* results—surely a striking advance.

The bulk of the book (pages 84–257) is a table giving the minimum decompositions of all integers from 3,470,000 to 3,600,000 into a sum of (positive) fifth powers of integers. Letters a, · · · , B are used to denote the fifth powers of 2, · · · , 20, making possible the printing of the table in four columns to the page. Thus, under the heading 3470 we find opposite 19 the entry 3dfhjmB, which states that 347019 is the sum of the fifth powers of 3 and the numbers denoted by d, f, h, j, m, B. To eliminate typographical errors (and possibly also to keep down expense), the entire book, with the exception of the preface, has been reproduced without typesetting or linotype from the original typescript.

E. T. Bell

*Synthetische Geometrie.* By H. Liebmann. Teubner, Leipzig, 1934. viii+119 pp.

This is volume 40 of Teubner's Mathematische Leitfäden written by Professor Liebmann of the University of Heidelberg. It is an elementary treatise on projective geometry, based on strictly axiomatic foundations implied by modern research and criticism, and is a valuable aid to students and teachers of this branch of mathematics.

Arnold Emch

# NOTES

The Association for Symbolic Logic is now publishing a quarterly journal entitled The Journal of Symbolic Logic, the first number of which appeared in May. The editors are Professors Alonzo Church, of Princeton University, and C. H. Langford, of the University of Michigan.

The American Association for the Advancement of Science has appointed Professor Oswald Veblen, of the Institute for Advanced Study, as its representative at the International Congress of Mathematicians at Oslo, July 13–18.

The regular spring meeting of the American Physical Society was held in Washington, April 30–May 2, following the meeting of the National Academy of Sciences.

The National Academy of Sciences held its annual meeting in Washington, April 27–29. Four papers in mathematics were presented. Professor Edward Kasner, of Columbia University, spoke on *Conformal and equilong symmetry*; Professor Marston Morse, of the Institute for Advanced Study, on *Abstract equilibrium theory*; Professor G. D. Birkhoff, of Harvard University, on *The problem of stability in dynamics*; and Professor Dunham Jackson, of the University of Minnesota, on *Problems of closest approximation in two variables.*

The Joseph Henry Lecture of the Philosophical Society of Washington was given on February 1, 1936, by Dr. Herbert Dingle, assistant professor of astrophysics at the Imperial College of Science and Technology, London. His subject was *The physical universe.*

Professor Oskar Bolza of the University of Freiburg, who taught most inspiringly in this country at Johns Hopkins University, Clark University, and the University of Chicago, took his doctor's degree at the University of Göttingen on June 28, 1886, fifty years ago. Congratulations were forwarded to him by the Deutsche Mathematiker-Vereinigung, by the American Mathematical Society, and doubtless by other learned societies. Many of his former students in this country have taken occasion to write appreciatively to him at this time.

On Thursday evening May 14, Dr. Henry Seeley White, professor emeritus of mathematics at Vassar College and past president of this Society, was honored by a dinner given by members of the Vassar faculty. About ninety guests were present at the dinner. Professor White has been chairman of the department of mathematics since 1905, and, although he became professor emeritus in 1933, the trustees have regarded his services so highly that for these last three years they have retained him as Senior Lecturer. His work as teacher ends in June, 1936, but his friends know that for him leisure will not mean inactivity but more time for the problems on which he has been working for years. President H. N. MacCracken as toastmaster introduced the following after dinner speakers: Dr. Grace Macurdy of the department of Greek; Dr. H.

E. Mills, professor emeritus of economics and a former colleague of professor White; Dr. T. S. Fiske, professor emeritus of mathematics at Columbia University and chairman of the College Entrance Examination Board; Dr. H. S. Taylor, professor of chemistry at Princeton University; and Dr. M. E. Wells, professor of mathematics at Vassar.

Dean R. G. D. Richardson, of Brown University, Secretary of this Society, has prepared a careful and interesting report on the doctor's degree and mathematical research, published in the April American Mathematical Monthly, which will be of great service to American mathematicians. This report and the data available in Dean Richardson's office contain such fundamental statistical information as the names of all Americans who have received the doctorate in mathematics, the names of foreign mathematicians who have joined our American mathematical group, the research papers presented to the Society and subsequently published by each of these persons, and the names of all teachers of mathematics in American universities, colleges, normal schools, and junior colleges during the year 1935–36.

The Xavier Heuschling statistics prize of the Royal Academy of Belgium has been awarded jointly to Léon-H. Dupriez, of Louvain, and to P. Bourgeois and J. F. Cox, of the University of Brussels.

Franklin Medals of the Franklin Institute have been awarded to Dr. F. B. Jewett, of the American Telephone and Telegraph Company and the Bell Telephone Laboratories, and to Dr. C. F. Kettering, of the General Motors Corporation.

The following have been elected members of the Royal Society of London: Dr. A. C. Aitken, of the University of Edinburgh, and Professor N. F. Mott, of the University of Bristol.

Dr. Peter Debye, director of the Physical Institute of the University of Leipzig, recently appointed professor of physics at the University of Berlin, was elected a member of the American Philosophical Society at the annual meeting held in Philadelphia, April 23–25.

The honorary degree of doctor of laws was conferred on Professor J. E. Littlewood, Rouse Ball professor of mathematics at the University of Cambridge, by the University of St. Andrews.

Guggenheim Fellowships have been awarded to Dr. Solomon Gandz, of New York City, and to Professor M. H. Stone, of Harvard University. Dr. Gandz will study early Arabic algebra, especially its connection with ancient Greek, Babylonian, and Egyptian mathematics, and its influence upon mediaeval European mathematics. Professor Stone will carry on research in the field of the theory of linear representation in abstract space.

The following have been appointed National Research Fellows in Mathematics for the year 1936–37: D. N. Dribin, Aaron Fialkow, Nathan Jacobson, Norman Levinson, and C. B. Tompkins. This list includes renewals.

Professor A. L. Candy, of the University of Nebraska, has been named professor emeritus at that institution.

Professor H. B. Curry, of Pennsylvania State College, has recently been made vice-president of the Association for Symbolic Logic.

At the centenary celebration of the University of London, an honorary degree was conferred on Professor Albert Einstein, of the Institute for Advanced Study.

The honorary degree of doctor of science was conferred on Professor E. R. Hedrick, of the University of California at Los Angeles, by the University of Michigan on June 20.

Professor Solomon Lefschetz, of Princeton University, gave three lectures on *Combinatory topology* at the University of Paris in May.

Dr. J. A. Lyon, professor of physics at Newcomb College, Tulane University, has been elected president of the Astronomical Society of New Orleans.

Professor Marston Morse, of the Institute for Advanced Study, was elected a member of the American Philosophical Society at the annual meeting held in Philadelphia, April 23–25.

Professor J. L. Walsh, of Harvard University, was elected a member of the National Academy of Sciences at the annual meeting in Washington, April 27–29.

Professor O. Eggert, of the Berlin Technical School, has been appointed director of the Geodetic Institute at Potsdam.

Dr. Enrico Fermi, professor of theoretical physics at the University of Rome, and member of the Royal Academy of Italy, will become a member of the faculty of the summer school of Columbia University.

Dr. Otto Oesterhelt, of the Dresden Technical School, has been promoted to a professorship in mathematics.

Professor J. G. Semple, professor of mathematics in the Queen's University, Belfast, has been appointed professor of mathematics at King's College, London.

Dr. Bengt Stromgren, lecturer on astrophysics at the University of Copenhagen, has been appointed assistant professor of theoretical astrophysics at the University of Chicago.

Dr. Felix Bernstein, of Columbia University, has been appointed to a professorship in biometrics at New York University.

Associate Professor H. C. Carver, of the University of Michigan, has been promoted to a professorship.

Dr. S. Chandrasekhar, of Madras, has been appointed research associate in mathematical astronomy at the University of Chicago.

Mr. D. J. Colbert, of St. Bonaventure College, has been promoted to an assistant professorship.

Assistant Professor C. C. Craig, of the University of Michigan, has been promoted to an associate professorship in mathematics.

Dr. Ben Dushnik, of the University of Michigan, has been promoted to an assistant professorship.

Professor W. B. Fite has been appointed to the newly established Davies professorship at Columbia University.

Dr. C. A. Garabedian has been appointed to an associate professorship of mathematics at Wheaton College.

The retirement of Dr. Robert Henderson, Vice-President of the Equitable Life Insurance Company, is announced.

President I. O. Horsfall, of Snow College, Utah, has been appointed director of the Extension Division of the University of Utah.

Professor B. O. Koopman, of Columbia University, will teach at Harvard University during the next academic year.

Professor R. E. Langer, of the University of Wisconsin, is teaching in the present summer session at Stanford University.

Associate Professor R. B. Lindsay, of Brown University, has been promoted to a professorship in theoretical physics and has been appointed chairman of the department of physics there.

Dr. E. W. Miller, of the University of Michigan, has been promoted to an assistant professorship.

Professor R. E. Moritz has retired from his position as head of the department of mathematics at the University of Washington, but will continue his teaching there. He will be succeeded by Professor A. F. Carpenter.

Professor W. V. Parker, of the Georgia School of Technology, has been appointed to an associate professorship in mathematics at the Louisiana State University.

Dr. E. B. Roessler, of the University of California at Davis, has been promoted to an assistant professorship in mathematics and appointed assistant statistician in the Experiment Station there.

Associate Professor D. M. Smith, of the Georgia School of Technology, has been promoted to a professorship and has been appointed head of the department of mathematics.

Dr. H. E. Spencer, of Cornell University, has been appointed to a professorship at Presbyterian College, South Carolina.

Assistant Professor M. E. Stark, of Wellesley College, has been promoted to an associate professorship.

Mr. John Oxtaby and Dr. Stanislaw Ulam have been elected to the Society of Fellows of Harvard University.

Dr. Max Zorn, of Yale University, has been appointed to an associate professorship at the University of California at Los Angeles.

The following appointments to instructorships in mathematics are announced: Brooklyn College: Dr. James Singer; University of California at Los Angeles: Dr. M. R. Hestenes; Colby College: Dr. I. J. Schoenberg; Cornell University: Dr. J. H. Curtiss, Dr. Saunders MacLane, Dr. J. B. Rosser, Dr. R. J. Walker; Georgia School of Technology: Dr. W. E. Sewell; Harvard University: Mr. R. P. Boas, Dr. A. E. Pitcher (Benjamin Peirce Instructor); Massachusetts Institute of Technology: Dr. W. T. Martin; University of Michigan: Dr. Ralph Hull; Mississippi State College: Dr. Arthur Ollivier; New York State Teachers College: Dr. C. Grace Shover; Pomona College: Dr. H. J. Hamilton; University of Washington: Dr. A. H. Taub.

Emeritus Professor A. C. Dixon, of Queen's University, Belfast, died on May 4, 1936, at the age of seventy-one years.

The death of Professor Alf Guldberg, of the University of Oslo, is announced.

Professor Gustave Juvet, of the University of Lausanne, died on April 7, 1936.

The death of Professor D. Mercogliano, of Naples, is reported.

Dr. Karl Pearson, emeritus professor of eugenics and formerly director of the Francis Galton Laboratory for National Eugenics at the University of London, died on April 27, 1936, at the age of seventy-nine years.

Professor I. L. Miller, of South Dakota State College, died on February 22, 1936.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

**The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.**

256. Professor C. M. Cramlet: *Linear differential equations with constant coefficients.*

First order differential equations are solved by reducing the equations to canonical form by elementary matrix operations. The method is elementary and practical for the solution of the homogeneous and non-homogeneous systems. (Received May 7, 1936.)

257. Professor C. M. Cramlet: *On the reduction of a representation to classical canonical form.*

A representation or mixed tensor $a_s^t$ transforms in accordance with the matrix equations $\bar{a} = paq$, $pq = 1$. The transformation matrix $p$ such that $a$ is transformed to canonical form $\bar{a}$ is found by simultaneously choosing $p_i$ and $q_i$ as elementary matrices, transforming $a$ to canonical form by a succession of transformations. The method is practical for the reduction of a numerical matrix $a$. (Received May 7, 1936.)

258. Professor H. P. Robertson: *Geometry and physical space-time.*

There are two roles which geometry, or its four-dimensional extension, kinematics, may assume in physical science: (1) an *a priori* form, prerequisite to a quantitative description of the physical world, and (2), a *contingent* form, conditioned by the physical content of space-time. The following topics are treated: Influence on the former position, as exemplified by Newton and Kant, of the classical non-euclidean geometries of Gauss, Bolyai, Lobachévski, and Riemann; mathematical and philosophical development of this view by Helmholtz, Lie, and Russell. Extension to the four-dimensional space-time manifold in the special theory of relativity by Einstein and Minkowski, and in the later theories of de Sitter, Whitehead, and Milne. Development of the second position, envisaged by Riemann, in the general theory of relativity, by Einstein and Weyl; applications to cosmology by Friedmann, Lamaître, Tolman, and the author. Interfusion of the second position with the methods of the first in the cosmological applications of the author's kinematical investigations, and

the relationship thereto of the theories of Milne and Page. (Received May 7, 1936.)

259. Professor L. H. McFarlan: *Special forms of the Euler differential equations when x is absent from the integrand.*

In this paper the special form $f-y'f_{y'}=c$ assumed by the Euler differential equation for the most simple problem of the calculus of variations, when the integrand is free of $x$, is seen to follow in a direct manner if the problem is treated in the parametric form. It may also be considered as a problem of Lagrange with the initial end point fixed while the other end point is constrained to lie on a line parallel to the $t$ axis. The auxiliary differential equation is $x'-1=0$. The transversality condition at the variable end point yields the special form of the Euler equation. This method of treatment lends itself readily to problems in which derivatives of order higher than the first enter into the integrand. (Received May 11, 1936.)

260. Professor J. P. Ballantine: *Arc cotangent triads.*

An arc cotangent triad is defined as a set of three numbers, the sum of whose arc cotangents vanishes; for example, (1, 1, 0), (−1, 2, 3), (−2, 3, 7). If one such triad $(a, b, c)$ is known, then obviously $(-a, -b, -c)$, $(b, a, c)$, and less obviously $(-a, b+2a, c+2a)$ are also. This leads to an interesting group of transformations generated by five self-reciprocal operations. It is proved that all integral arc cotangent triads can be generated from (1, 1, 0) by means of this group. (Received May 15, 1936.)

261. Dr. F. B. Jones: *Concerning more general topologically flat spaces.*

Suppose that $S$ is a space satisfying R. L. Moore's axioms 0-4 and the following axiom: If $P$ is a point of a region $R$, there exists in $R$ a connected domain $D$ containing $P$ whose boundary is a subset of a collection $\Delta$ of mutually exclusive continua lying in $R-D$ such that no component of $D-P$ has a limit point in more than one element of $\Delta$. In this paper the author shows that a number of plane analysis situs theorems hold in $S$ and that if $S$ is completely separable, $S$ is a subset of a plane or a sphere. (Received May 18, 1936.)

262. Dr. F. B. Jones: *Certain equivalences and subsets of a plane.*

In a space satisfying R. L. Moore's axioms 0-4 and the author's axiom $\overline{5}_1$ (see abstract 39-9-254) a number of plane analysis situs theorems are shown to be equivalent. An example of a completely separable space satisfying these axioms is given in which none of these theorems hold true. This example also refutes the author's statement in the above abstract that such a completely separable space is a subset of a plane or a sphere. However, conditions are discussed which would make such a space a subset of a plane or a sphere. (Received May 18, 1936.)

263. Mr. W. H. Ingram: *On the forces occurring in a dynamical system where the coordinate axes have variable relative obliquity.* Preliminary report.

When the relative obliquity of a system of coordinate axes varies with the time, corresponding forces arise in the equations of motion of the dynamical system whose coordinates are referred to such axes. Application is made to the theory of commutator motors where the coordinate axes are rotating and, under certain conditions, mutually oblique with variable obliquity. (Received May 18, 1936.)

264. Professor L. I. Neikirk: *Some symbolic identities.*

Given an algebraic identity $\phi_1(y) = \phi_2(y)$; if $y$ is replaced by the differentiation symbol $D$, we derive an operational identity $\phi_1(D) = \phi_2(D)$. When this is applied to a second identity $F_1(x) = F_2(x)$, the result will be a third identity. Most identities obtained in this way are easily obtained by other methods. It has, however, yielded valuable results in differential equations, invariants and covariants, and theory of numbers. Among the cases are Cayley's hyperdeterminates, Aronhold's symbolic notation in invariants and covariants, and Blissard's umbral notation in the theory of numbers. The author uses the method to expand the inverse operator and obtains the solution of the general linear differential equation of the first order as an infinite series of iterated integrals which are reduced to the conventional form. (Received May 19, 1936.)

265. Dr. R. D. James: *An extension of two formulas of Hurwitz.*

There are two formulas due to Hurwitz for the number of representations of the square of an arbitrary integer as a sum of three or five squares. In this paper an analogous formula for seven squares is proved. In addition it is shown that, in certain cases, no further formulas of this type are possible. (Received May 19, 1936.)

266. Dr. Max Zorn: *On a theorem of Lie.*

Given a linear family $(x_1, \cdots, x_k)$ of matrices over a non-modular field such that (a) $xy - yx$ is in the family; (b) every matrix in the family is nilpotent; then the theorem of Lie asserts that every matrix in the associative ring generated by the $x_i$ is nilpotent, and that moreover any product of sufficient length in this ring vanishes. For this theorem a new proof is given in the following fashion: I. If symbols $x_i$ are commutative, then each product of $x_i$ is expressible as a sum of powers. II. The same is true if the $x_i$ have the property (a). III. Hence in the ring in question any element is a sum of nilpotent elements. IV. Such a ring is necessarily nilpotent. (Received May 20, 1936.)

267. Dr. Max Zorn: *Nilpotency of finite groups.*

If $a, b$ are elements of a finite group $G$, denote $aba^{-1}b^{-1}$ by $\langle a, b\rangle$. For two subgroups $U, V$ let $\langle U, V\rangle$ be the subgroup generated by the $\langle u, v\rangle$. We prove the theorem: If all iterated commutators $\langle a\langle a\langle a \cdots \langle ab\rangle\rangle\rangle\rangle$ of sufficient length

are identity, then the group $G$ is nilpotent, that is $\langle G\langle G\langle G \cdots \langle G, G\rangle\rangle\rangle\rangle$ is identity. The analog of this theorem in the theory of continuous groups is the theorem of Lie treated in the foregoing abstract. (Received May 20, 1936.)

268. Mr. I. E. Highberg: *A note on pseudo-polynomials in abstract spaces.*

In a previous communication (see abstract 42-5-136) I discussed the definition of pseudo-polynomials, gave a number of their properties, and by extension of a method due to Fréchet showed that in a complex space a pseudo-polynomial is a solution of the functional equation $\Delta^{n+1}f(x) \equiv 0$, where the increments are independent and the solution is assumed to be continuous. In this note a simpler derivation is given, depending on the results of Orliez and Mazur. Some additional properties of pseudo-polynomials are also considered. (Received May 21, 1936.)

269. Mr. Ivan Niven and Professor F. S. Nowlan: *A note on association by quartic matric polynomials.*

This paper extends results obtained by Dr. M. M. Flood, which were reported upon by him to this Society (abstract 39-3-92). Dr. Flood very kindly gave the authors access to his work. Dr. Flood's paper dealt with necessary and sufficient conditions for association by linear, quadratic, and cubic matrix polynomials. The present paper determines necessary and sufficient conditions for association by quartic matric polynomials. Unfortunately the relations obtained are extremely involved and do not suggest a solution of the general problem. (Received May 21, 1936.)

270. Mr. P. O. Bell: *Covariant configurations associated with a general curved surface at one of its points.*

In a recent paper the author completed the geometric characterization of the series of tetrahedrons associated with canonical developments for the equation of a general curved surface at one of its points. This fundamental characterization is used to determine certain configurations which are covariantly associated with the surface at one of its points. The geometry of these configurations is then studied with particular emphasis on the relations of these to previously known configurations. Among the configurations which are located and studied are cubic scrolls, twisted cubics, plane cubics, tricuspidal quartics, conics, and sequences of curves associated with a given covariant curve associated with the surface at one of its points. (Received May 21, 1936.)

271. Professor E. T. Bell: *Arithmetical consequences of a trigonometric identity.*

The formula for a product of sines leads at once to Tardy's identity (Annali di Scienze Mathematiche e Fisiche, vol. 2 (1851), pp. 287–291), and to others, from which numerous easier "Waring" problems can be seen by inspection. (Received June 9, 1936.)

272. Professor H. C. Ayres: *Existence and embedding theorems for hyperbolic systems of partial differential equations.*

A hyperbolic system of two first order partial differential equations in two independent variables and two unknown functions is studied. It is found that if such a system involves functions of class $C'''$ a unique solution can be found which takes on prescribed initial values on any sufficiently small non-characteristic arc. Also if such a system be of class $C^{(7)}$ and a solution of class $C^{(7)}$ be given it is found that the given solution can be embedded in a one parameter family of nearby solutions. (Received June 10, 1936.)

273. Mr. Garrett Birkhoff: *The enumeration of distributive lattices.*

It is shown that the different distributive lattices of dimensions $n$ [$n$ finite] correspond one-one to the different partially ordered systems $y$ of $n$ elements. The correspondence is $Y \rightleftarrows B^Y$, where $B$ denotes the Boolean algebra of two elements, and the definition of "powers" directly generalizes the usual notion of powers $m^n$ of cardinal numbers $m$ and $n$. (Received May 21, 1936.)

274. Professor Salomon Bochner: *A converse of Poisson's theorem in the theory of probability.*

If each of a large number of urns contains the same total number of white and black balls, and if on drawing one ball from every urn white appears $m$ times and black $n$ times, then the average number of white balls in all urns taken together has a probable value that can easily be computed in terms of $m/n$ and of the a priori probability concerning the unknown distribution of white and black in the different urns. (Received June 6, 1936.)

275. Dr. Joel Brenner: *The normal subgroups of the groups of matrices* mod $p^r$.

Let $G$ be the multiplicative group of all non-singular square matrices of side $n$, mod $p^r$. This is of course the group of automorphisms of the Abelian group of order $p^{nr}$ and type $(r, \cdots, r)$. The author has found all the normal subgroups of $G$, except in the case $p=2$. For a variable matrix $A=(a_{ij})$ of $G$, define $s(A)=\min_{ij}(a_{ij}, a_{ii}-a_{jj}, p^r)$. One can prove that $s(AB) \geqq \min(s(A), s(B)$, and $s(B\,AB)=s(A)$. The main lemma is: If a normal subgroup $N$ of $G$ contains any matrix $A_0$, then $N$ contains all matrices $A=(a_{rj})$ with $s(A) \geqq s(A)_0$ of determinant unity mod $p^r$ for which $a_{kk} \equiv 1 (p^{s(A_0)})$. Rules are given enumerating explicitly the normal subgroups, and stating exactly which matrices occur in each. Related results of a more miscellaneous nature are also given. (Received May 16, 1936.)

276. Professor W. C. Graustein: *Extensions of the four-vertex theorem.*

The four-vertex theorem states that an oval has at least four vertices, that is, that the curvature of an oval has at least four extrema. It is here shown that the theorem holds for larger classes of closed regular plane curves: (a) for

all simple curves; (b) for all curves with angular measure $\pm 2\pi$ or zero which have an arc of type $\Omega$; (c) for all curves with angular measure $\pm 2n\pi$, $n \geqq 2$, which have an arc of type $\Omega$ and points of inflection. By "angular measure" is meant the algebraic angle through which the directed tangent turns when the curve is completely traced. By an "arc of type $\Omega$" is meant a simple (open or closed) arc $AB$ of non-negative curvature which is tangent to the same line at $A$ and $B$ and lies on one side of this line. (Received June 13, 1936.)

277. Dr. Marshall Hall: *An isomorphism between linear recurring sequences and algebraic rings.*

This paper makes use of a fundamental isomorphism between linear recurring sequences and algebraic rings in order to investigate the arithmetic properties of the sequences. The pattern of the periods and numerics of sequences satisfying a fixed recurrence is investigated in detail. Null sequences are considered, and necessary and sufficient conditions are found that a sequence be null for a given modulus, and also that a sequence be $p$-adically null.

Fundamental theorems are proved on the distribution of residues in the period, and application of these theorems to certain special recurrences yields a curious set of diophantine equations which the distribution numbers must satisfy. (Received June 15, 1936.)

278. Professor Einar Hille: *A problem in Factorisatio Numerorum.*

A study is made of the function $f(n)$ which gives the number of representations of $n$ as a product of factors greater than one, two representations being considered identical if and only if they contain the same factors written in the same order. Estimates are given of $\sum f(n)$ if the summation is extended over all $n$ of the form $p_1^{a_1}p_2^{a_2} \cdots p_n^{a_n}$ where the sum of the exponents is a given integer and the primes are fixed or over all $n$ of the given form which are less than a given $N$, and so on. Estimates of $f(n)$ itself are obtained as a particular case. The essential tools are functional equations, generating power series and Dirichlet series, and the Ikehara-Wiener theorem. Extensions are obtained to the case of factorizations of ideals in algebraic fields and in certain congruence fields. (Received May 22, 1936.)

279. Professor G. M. Merriman: *Concerning polynomials simultaneously orthogonal on more than one curve.*

Szegö has recently exhibited (Transactions of this Society, vol. 37 (1935), pp. 196–206) a complete list of the polynomials $p_n(z)$ in a complex variable which are simultaneously orthogonal with respect to suitable norm functions on all the level curves of a given family. In the present paper it is shown that if the same set of polynomials is simultaneously orthogonal (with respect to an arbitrary norm function) on more than one of a family of level circles or of level ellipses, then the set is orthogonal on all the level curves of the family. (Received May 27, 1936.)

280. Professor Oystein Ore: *On the theorem of Jordan-Hölder.*

This paper contains an analysis of the theorem of Jordan-Hölder for arbitary structures. The former investigations on the theorem of Jordan-Hölder in structures satisfying the Dedekind axiom (Dedekind, Garrett Birkhoff, Ore) correspond to the case of principal series and not to composition series in groups. By introducing normal elements in an arbitrary structure it is possible to derive a theorem which in many ways is analogous to the general theorem of Schreier-Zassenhaus for groups. The analogy is, however, not complete. (Received May 22, 1936.)

281. Mr. J. C. Oxtaby: *The category and Borel class of certain subsets of $L_p$.*

The subsets of $L_p([0, 1])$, $p \geqq 1$, corresponding to continuous functions, Riemann integrable functions, upper and lower semicontinuous functions, are shown to be $F_{\sigma\delta}$ sets of first category. The set $L_qCL_p$ for $q>p$ is shown to be $F_\sigma$ and likewise of first category. The method consists in obtaining a representation for each set in terms of closed sets, utilizing boundedness properties of the functions. (Received May 28, 1936.)

282. Dr. S. Saks: *On some functionals.* II.

This paper contains some extensions of and corrections to, the results obtained in a previous paper published in the Transactions under the same title. (Received June 15, 1936.)

283. Professor M. H. Stone: *Applications of Boolean algebras to general topology.*

This paper carries the author's treatment of general topology by the use of Boolean algebras somewhat further. In addition to results previously announced in this Bulletin and in the Proceedings of the National Academy, it offers a discussion of the maps of completely regular spaces in Boolean spaces. This discussion involves the analysis of the relations between the algebraico-topological structure of the class of all bounded continuous functions on a topological space and the topological structure of the underlying space. The existence of bicompact Hausdorff extensions of a completely regular space is discussed in detail. (Received June 5, 1936.)

284. Professor J. L. Walsh and Mr. W. E. Sewell: *Note on degree of approximation to an integral by Riemann sums.*

Let $f(x)$ be defined and integrable Riemann for $a \leqq x \leqq b$. The authors investigate here the degree of approximation to the Riemann integral by the corresponding Riemann sum when equidistant ordinates are used. They consider continuous functions, functions of bounded variation, functions continuous except for finite jumps, and functions for which there are approximating polynomials converging with a given degree. A typical result is: *Let the function $f(x)$ be defined in the interval $0 \leqq x \leqq 2\pi$ and have the property that the trigonometric polynomials $S_N(x) = \sum_{n=0}^{n=N} (a_{Nn} \cos nx + b_{Nn} \sin nx)$ of respective orders $N=0, 1, 2, \cdots$, exist such that $|f(x) - S_N(x)| \leqq \epsilon_N$. Then one has $\left|\int_0^{2\pi} f(x)dx - (2\pi/N)\sum_{k=1}^{N} f(2k\pi/N)\right| \leqq 4\pi\epsilon_{N-1}$, $N>0$.* (Received May 10, 1936.)

285. Dr. Rufus Oldenburger: *Rational equivalence of a form to a sum of pth powers.* Preliminary report.

In a paper in the 1933 Proceedings of the Cambridge Philosophical Society Bronowski translated into geometry the problem of the equivalence of a $p$-ic to $y_1^p+\cdots+y_r^p+\lambda_1 v_1+\cdots+\lambda_s v_s^p$. The linear transformations used by Bronowski were not non-singular, and the translated problem does not give a solution of the algebraic problem. The present paper solves the problem of the equivalence in a given field of a general $p$-ic form to the canonical form $ax^p+\cdots+cw^p$. The problem is treated with the aid of the author's theory of non-singular multilinear forms to appear in the forthcoming issue of the Transactions of this Society, and arithmetic invariants, which have been studied by Hitchcock and the author. The arithmetic invariant method of approach is a novel one in the theory of forms of degree higher than two. Also, the attainment of results for general $p$-ic forms, $p\geqq 3$, by treating these as special cases of multilinear forms is new. (Received March 6, 1936.)

286. Mr. Garrett Birkhoff: *Lie groups simply isomorphic with no linear group.*

It is shown that there exists a three-parameter Lie group, simply isomorphic even in the purely algebraic sense with no linear group. Further, every "hypercentral" Lie group of matrices is locally analytically isomorphic with a Lie group having the same defect. This contains the result announced in abstract 42-5-240. (Received June 8, 1936.)

287. Professor R. V. Churchill: *On an expansion of the inverse Laplace transformation and its application.*

The success of the Laplace transformation method of solving boundary problems in partial differential equations depends largely upon the possibility of finding the inverse transformation of the known function which appears as the solution of the transformed problem. In this paper conditions on the function are found which are sufficient to permit an expansion of its inverse transformation in a convergent series. This series is a generalization of the Heaviside partial fractions expansion. The new conditions are found with the aid of those of Tamarkin (Transactions of this Society, vol. 28 (1926), pp. 417–425) for the solution of the Laplace integral equation in the form of a complex integral. The use of the series and the conditions is illustrated by solving a problem in the forced longitudinal vibrations of a prismatic bar. (Received June 30, 1936.)

288. Professor J. L. Doob: *Stochastic processes depending on a continuously varying parameter.*

Let $\Omega^*$ be the space of all functions $x(t)$ defined for $-\infty<t<\infty$. It has been shown by Kolmogoroff that a non-negative completely additive function of sets, $P^*(A^*)$ can be defined on the Borel system of sets determined by the sets of elements $x(t)$ satisfying conditions of the form $a_j<x(t_j)<b_j$, $j=1,\cdots,n$. A set in this Borel system, or one which differs from such a set by a subset of a

set for which $P^*$ vanishes, will be called $P^*$-measurable. The exterior $P^*$-measure of any subset of $\Omega^*$ is defined as the greatest lower bound of the $P^*$-measures of the $P^*$-measurable sets which contain it. If $\Omega$ is a subset of $\Omega^*$ of exterior $P^*$-measure 1 (we are supposing that $P^*(\Omega^*)=1$) we define a measure $P(\Lambda)$ for any set $\Lambda$ which is the intersection of $\Omega$ with a $P^*$-measurable set $\Lambda^*$: $P(\Lambda)=P(\Lambda^*)$. The conjunction of such a space $\Omega$ with the measure function $P(\Lambda)$ is defined as a stochastic process. For each value of $t$, $x(t)$ can be considered as a measurable function on $\Omega$, that is, a chance variable. Stochastic processes have been previously defined as one-parameter families of chance variables. The present definition thus includes the old one, but also makes possible an examination of the characteristics of the functions $x(t)$ in $t$, such as continuity properties, for example. (Received June 22, 1936.)

289. Professor W. L. Duren: *A problem of Zermelo in the calculus of variations.*

The problem of minimizing the integral $\int_{x_1}^{x_2} f(x, y, y', \cdots, y^{(n)})dx$ in a class of arcs $y=y(x)$ joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is usually treated by transforming it into a problem of Lagrange. An arc $E$ may be said to furnish a relative minimum of order $K$ if it minimizes the integral with respect to arcs having elements $(x, y, y', \cdots, y^{(K)})$ in a neighborhood of those of $E$. This paper presents sufficient conditions for relative minima of order $K(K=0, 1, \cdots, n)$ whereas the general theory of the Lagrange problem furnishes no information about relative minima of order less than $n-1$. (Received July 1, 1936.)

290. Professor Philip Franklin: *The four color problem.*

In 1920 the author gave certain reductions for the four color problem. By combining these with earlier ones due to Kempe and Birkhoff he showed that a map, not colorable in four colors, must contain at least 26 regions. In 1926 Reynolds succeeded in increasing this number to 28. His argument was simplified by the use of certain results due to Errera who in 1924 proved that every uncolorable map must contain at least 13 pentagons. In the present paper a number of new reducible configurations are treated. Some of these, involving pentagons and heptagons, were suggested by their appearance in simple maps devoid of previously known reducible configurations. By using these new reductions, the following two theorems are proved: Every uncolorable map contains at least 15 pentagons. Every uncolorable map contains at least 32 regions. (Received June 30, 1936.)

291. Dr. Aline H. Frink: *General distance functions and the metrization problem.*

In proving that under certain conditions a topological space is metrizable, a distance function is often introduced satisfying, instead of the triangle axiom, the condition: If $ab<\epsilon$, and $cb<\epsilon$, then $ac<2\epsilon$. It is shown that if this condition holds, a metric satisfying the triangle axiom can be defined directly, without making use of Chittenden's theorem which says that a space with a uniformly regular distance function is metrizable. The method also leads to a much simpler proof of Chittenden's theorem itself, since it does not depend on

the result that a normal space has non-constant continuous functions. For the case of the metrization conditions of Alexandroff and Urysohn, Niemytzki, and W. A. Wilson, it is shown that a triangle axiom metric can be introduced directly in terms of the original conditions. New metrization conditions are also given for spaces having an unsymmetric distance function. (Received June 25, 1936.)

292. Professor Orrin Frink: *Geodesic continua in abstract metric space.*

Menger has given a definition of geodesic arcs in abstract metric space, and has proved an existence theorem for them. By extending the definition to geodesic continua, and proving that the Carathéodory linear measure function is lower semicontinuous for continua, existence theorems for geodesics are obtained with very general end conditions, including conditions analogous to those of Morse in *The Calculus of Variations in the Large*. In the special case where there are just two end conditions, the geodesic continua obtained are shown to be arcs. In addition to the great generality of the end conditions and of the underlying space, the Carathéodory linear measure function used is general enough to include many of the functionals of the calculus of variations defined in terms of a single integral. (Received June 25, 1936.)

293. Dr. Max Herzberger: *A general theorem for rotationally symmetrical optical systems.*

The author demonstrates a new simple theorem, which holds true for skew rays in rotation symmetric optical instruments, and draws from it several interesting conclusions. One of them is the existence of a kind of Gaussian optics on skew rays, another is, for instance, the fact that these equations allow an integration under certain conditions, and so allow us to determine the law which must hold true in a perfect photographic or microscopic objective. (Received July 1, 1936.)

294. Professor E. V. Huntington: *Postulates for assertion, conjunction, negation, and equality.*

The purpose of this paper is to present an "abstract" mathematical theory so constructed that one of its "concrete interpretations" shall be the calculus of propositions in ordinary logic. The "base" here selected is $(K, T, \times, ', \equiv)$, where $K$ is a class of elements $a, b, c, \cdots$ (interpretable as "propositions" or "sentences"); $T$ is a subclass (interpretable as "asserted" propositions); $ab$ (read $a$ times $b$) is the result of a binary operation (interpretable as the "conjunction" of $a$ and $b$); $a'$ (read $a$ dash) is the result of a unary operation (interpretable as the "contradictory" of $a$); and $a \equiv b$ (read $a$ quad $b$) is the result of another binary operation (interpretable as the "equalization" of $a$ and $b$). By definition, "$a = b$" means "$a \equiv b$ is in $T$"; the properties of "$=$" are not presupposed but are derived from the postulates on the selected base. The distinction between effective implication $[ab \equiv a]$ and Lewis's strict implication $[ab' \equiv Z$ (where $Z = aa')]$ is clearly analyzed; and both are contrasted with Russell's material implication $[(ab')']$. Proofs are given in full; no previous

acquaintance with Boolean algebra is required. (An abstract of part of the paper is given in the Proceedings of the National Academy of Sciences, May, 1936.) (Received June 30, 1936.)

295. Professor M. H. Ingraham and Dr. Margarete C. Wolf: *Relative linear sets and similarity of matrices whose elements belong to a division algebra.*

Given an $n \times n$ matrix $M$ with elements in a quasi-field, a polynomial $g = \sum \lambda^i a_i$ and an $n \times 1$ vector $\xi$, $g(M) \cdot \xi$ is defined to be the vector $\sum M^i \xi a_i$. If the $a_i$'s belong to a field, this is ordinary multiplication of $\xi$ by $g(M)$. A set of vectors $\xi_1 \cdots \xi_k$ is said to be linearly independent relative to $M$ if $\xi_i \neq 0 (i)$ and if $\sum g_i(M) \cdot \xi_i = 0$ implies $g_i(M) \cdot \xi_i = 0 (i)$. A set of vectors is a linear set relative to $M$ if the sum of every two vectors is in the set and if $g(M) \cdot \xi$ is in the set for every vector $\xi$ of the set and every polynomial $g$. A theory of relative linear sets is given yielding proper basis, minimal $g$'s such that $g(M) \cdot \xi = 0$, and so on. From this is readily deduced a complete rational theory of the similarity of matrices whose elements belong to a division algebra. As a special case a simple rational treatment of the ordinary theory of similarity of matrices with elements in a field is obtained. (Received June 29, 1936.)

296. Mr. J. Q. Jordan and Dr. Walter Leighton: *On convergence of continued fractions.*

Let (1) $[1 + K_1^{\infty}(x_n/1)]$ be a continued fraction in which the $x_n$ are arbitrary complex numbers $\neq 0$. This paper establishes the transformations on the $x_n$ which permute the convergents of (1) in sets of three and in sets of four. These results are used to establish many new simple convergence criteria for (1) of which a characteristic example is the following. The continued fraction (1) converges if $|x_{4n+1}| \leqq \frac{1}{4}$, $|x_{4n+2}| \geqq 3$, $|x_{4n+3}| \geqq 4$, $|x_{4n+4}| \leqq \frac{1}{4}$, $(n = 0, 1, 2, \cdots)$. (Received June 24, 1936.)

297. Professor Walther Mayer and Professor T. Y. Thomas: *Analytic arcs in analytic manifolds.*

It is proved that any analytic arc $C(t)$ of an analytic manifold $\mathfrak{M}$ can be embedded in an $n$-parameter family of analytic arcs $C(t, a)$. This result and a lemma on the continuation of an analytic function over an open set containing an analytic arc in $\mathfrak{M}$ are used to show that the set of all points $Q$ which can be joined to an arbitrary point $P$ of $\mathfrak{M}$ is both open and closed. As this set is non-vacuous it follows that any two points of $\mathfrak{M}$ can be joined by an analytic arc. The methods employed are such that this result admits an immediate extension to any finite number of points of the analytic manifold. (Received June 26, 1936.)

298. Mr. John Riordan: *Recurrence relations for the moments of Bernoulli and Poisson frequency distributions.*

The mixed difference recursion formulas for moments about the origin and the mean of Bernoulli and Poisson frequency distributions, given by A. T. Craig (this Bulletin, vol. 40 (1934), pp. 262–264), are shown to have power-

series solutions, the coefficients of which satisfy partial-difference recurrence relations. For the Bernoulli moment about the origin and the Poisson moments of both kinds, the relations are relatively simple. For the Bernoulli moment about the mean, the solution seems most simply expressed in terms of coefficients satisfying a pair of partial difference equations. Triangular arrays for the coefficients are given, covering the first ten moments about the origin and the first nine about the mean. (Received June 24, 1936.)

299. Dr. J. B. Rosser: *Extensions of some theorems of Gödel and Church.*

We shall say that a formal logic is simply consistent if there is no formula $A$ such that both $A$ and $\sim A$ are provable. It is proved for a large class of formal logics (roughly characterized by the phrase "adequate for elementary number theory") that: (1) Simple consistency implies that the logic is not adequate to prove its own consistency. (2) Simple consistency implies the existence of a formula $B$ such that neither $B$ nor $\sim B$ is provable. (3) Simple consistency implies the absence of any *Entscheidungsverfahren* for the logic. (2) is an extension of a theorem of Gödel and (3) an extension of a theorem of Church in having "simple consistency" in the hypothesis instead of "$\omega$-consistency." (1) is the well known Gödel theorem, but is proved for a larger class of logics. (Received June 18, 1936.)

300. Mr. W. E. Sewell: *A note on approximation by polynomials in the sense of least pth powers in the complex domain.*

The results of Dunham Jackson (this Bulletin, vol. 36 (1930), pp. 851–857; vol. 37 (1931), pp. 883–890) on approximation in the sense of least $p$th powers are sharpened in the case of a region bounded by a Jordan curve with corners, and extended to include a region bounded by an arbitrary rectifiable Jordan curve. (Received June 22, 1936.)

301. Professor T. Y. Thomas: *On simple analytic arcs.*

In this paper a number of lemmas are proved on the basis of which the following theorem is obtained: Any finite number of distinct points $P_1, \cdots, P_r$ of a connected analytic manifold of dimensionality $n \geqq 2$ can be joined by a simple analytic arc $C(t)$, where $0 \leqq t \leqq 1$, such that as $t$ increases from 0 to 1 the arc passes through the points in the given order. (Received June 26, 1936.)

302. Professor T. Y. Thomas: *On the covering of analytic arcs by coordinate neighborhoods.*

A coordinate neighborhood of an analytic manifold $\mathfrak{M}$ is said to be proper if it is homeomorphic to the interior of an $n$-dimensional euclidean sphere. It is said to be improper if it is merely the continuous map of a (proper) coordinate neighborhood of the $n$-dimensional euclidean space. The following two theorems are proved. *First*, any analytic arc $C(t)$ where $0 \leqq t \leqq 1$ is contained in an improper coordinate neighborhood $\Delta$ in $\mathfrak{M}$, the coordinates $y^1, \cdots, y^n$ of $\Delta$ being such that $-a < y^1 < 1+a$, $|y^j| < b$ for $j = 2, \cdots, n$, where $a$ and $b$ are sufficiently small positive constants, and such that the arc $C(t)$ is the portion

$0 \leqq y^1 \leqq 1$ of the $y^1$ axis. *Second*, any simple analytic arc $C(t)$, where $0 \leqq t \leqq 1$ in an analytic manifold $\mathfrak{M}$, admits a proper coordinate neighborhood $\vartheta$ with coordinates $y^1, \cdots, y^n$ such that $-\lambda < y^1 < \lambda + 1$, $|y^j| < \lambda$ for $j = 2, \cdots, n$, where $\lambda$ is a positive constant, and such that with respect to the coordinates $y^1, \cdots, y^n$ in $\vartheta$ the arc $C$ is the portion $0 \leqq y^1 \leqq 1$ of the $y^1$ axis. (Received June 26, 1936.)

303. Professor T. Y. Thomas: *Fields of parallel vectors in the large.*

Let $\mathfrak{M}$ be a simply connected analytic manifold over which an affine connection $\Gamma$ is defined (affinely connected space). The following theorem is proved: The space $\mathfrak{M}$ will admit $K$ (or more) independent fields of parallel contravariant vectors with components which are analytic functions of the coordinates of the neighborhoods of $\mathfrak{M}$ if, and only if, a certain polynomial $R_K(B)$ vanishes over $\mathfrak{M}$. Here $R_K(B)$ is a polynomial in the components of the curvature tensor $B$ and its first $n$ covariant derivatives, where $n$ is the dimensionality of the space. An analogous result of course holds for fields of parallel covariant vectors. (Received June 26, 1936.)

304. Professor T. Y. Thomas: *Riemann spaces of class one and their characterization.*

If necessary and sufficient conditions for the existence of a property $P$ of a (real) space are given by conditions of the form $F_1 = 0$, $F_2 \neq 0$, $F_3 > 0$, $F_4 \geqq 0$ where the $F$'s represent polynomials in the structural functions of the space and their derivatives to a specified order, these conditions are said to constitute an algebraic characterization of the property $P$. This paper deals with the algebraic characterizations of Riemann spaces as spaces of class one. In connection with the solution of this problem we have defined an integer invariant of a Riemann space which we have called the type number of the space. If the type number is one, the space is flat and hence fails to be of class one. We have excluded those Riemann spaces of type number two in as much as the discussion of such spaces requires essentially different methods than those of higher type number, and it has therefore been thought best to make these spaces the occasion of a separate investigation. For all other cases the algebraic characterizations have been constructed. (Received June 26, 1936.)

305. Professor G. T. Whyburn: *On continua of condensation.*

In this paper it is shown that a necessary and sufficient condition for a compact metric continuum $M$ to have property $N$ in the sense of R. L. Moore (see The Rice Institute Pamphlet, vol. 23, no. 1 (1936), p. 67, or this Bulletin, vol. 42 (1926), p. 35) is that $M$ be locally connected and that no cyclic element of $M$ have a continuum of condensation. (Received June 29, 1936.)

306. Professor G. T. Whyburn: *On boundaries in the plane.*

Let $S$ be a compact locally connected plane continuum. The property of being a boundary point of a complementary domain of $S$ is not invariant, of course, under all topological transformations of $S$ into plane continua, whereas

the property of being a boundary point of two or more such domains is invariant under such transformations. However, it is shown in this paper that if we suppose that the boundaries of no two complementary domains of $S$ intersect, then not only the property of being a point on the boundary of a complementary domain of $S$ but even the property of *being the boundary* of a complementary domain of $S$ is invariant under all topological transformations of $S$ into plane continua. (Received June 29, 1936.)

307. Professor G. T. Whyburn: *Semi-closed sets and collections.*

A set $K$ in a metric space $S$ is said to be semi-closed if each component of $K$ is closed and every convergent sequence of components of $K$ whose limit intersects $S-K$ converges to a single point. Similarly, a collection $G$ of disjoint sets is semi-closed if each set is closed and every convergent sequence of the sets whose limit intersects $S-G$ converges to a single point. In addition to proving elementary properties of such sets and collections relating them to upper semi-continuous collections, it is shown that if $F$ is a semi-closed subset of a compact locally connected continuum $S$ such that no component of $F$ separates $S$ locally and for any irreducible cutting $K$ of $S$ locally and for any irreducible cutting $K$ of $S$ between two points $K \cdot F$ is contained either in a countable number of components of $K$ or in a countable number of components of $F$, then $S-F$ is connected and locally connected. Various applications of this theorem to particular cases, such as that in which $S$ is unicoherent or is a sphere and $F$ is the sum of the boundaries (assumed disjoint) of the complementary domains of a continuum in $S$, are given. (Received June 29, 1936.)

308. Professor R. L. Wilder: *Locally connected subsets of euclidean n-space.*

This paper is an investigation of the external topological properties of closed subsets $J^k$ of euclidean $n$-space $E_n (n>2)$ with a view to generalizing the work of Schoenflies on continuous curves in the plane. (A set $J^k$ is a locally compact metric space that is locally $i$-connected for $i \leqq k$.) At all points $a$, the local Betti numbers $p^i(a, J^k)$ of a $J^k$ are $\leqq \omega$ for $i \leqq k$, and if in addition it is semi-$(k+1)$-connected, then $p^{k+1}(a, M) \leqq \omega$. If a semi-$(n-2)$-connected $J^{n-2}$ is embedded in $E_n$, then it is regularly $s$-accessible for $s>0$; and if it is a $J^{n-2}$ and $D$ is a domain complementary to it, the boundary of $D$ is regularly 0-accessible from $D$. A compact continuum in $E_n$ is a $J^0$ if the diameters of its complementary domains form a null sequence and the boundaries of its complementary domains are regularly $s$-accessible ($s \leqq n-2$) from these domains. Characterizations are obtained, in terms of external topological properties, for sets $J^k$ for various values of $k$. In case $k=0$ a characterization of the Jordan continuum in $E_n$ is obtained in terms of the properties of the bounding $(n-2)$-cycles of the complement. (Received June 26, 1936.)

309. Professor R. M. Winger: *Non-trochoidal rational curves of order n which admit dihedral groups of order $2n-4$.*

In a former paper (American Mathematical Monthly, vol. 39 (1932), pp. 578–589) projective trochoids of order $n$ invariant under dihedral collineation groups of orders $2n-2$ and $2n-4$ were discussed. Such curves are necessarily of even order. For all values of $n$ ($n>3$), even or odd, however, there is a one-parameter family of non-trochoidal curves invariant under a dihedral collineation group of order $2n-4$. These form the subject of the present study. The curves in question divide broadly into two classes, according as $n$ is odd or even, corresponding to the two main classes of dihedral groups. The family includes as special cases many remarkable individuals, among which may be mentioned the self-dual curves of Wear (American Journal of Mathematics, vol. 51 (1929), pp. 482–490), which are autopolar with respect to the maximum number of conics; and certain of the polar tangent curves treated by Stratton (Dissertation, University of Washington). In a metrical setting the curves are symmetric with respect to $n-2$ axes, equispaced about a point which is also a center of symmetry when $n$ is even. In this form the self-dual curves become the symmetric curves studied by Duncan (for $n$ odd) (this Bulletin, vol. 40 (1934), pp. 344–352). (Received June 29, 1936.)

## 310. Mr. C. B. Allendoerfer: *Einstein spaces of class one.*

This paper obtains necessary and sufficient conditions that an $n$-dimensional Einstein space ($n>3$) be of class one; that is, that it be not flat and that it admit an imbedding in an $(n+1)$-dimensional flat space. The conditions obtained consist of (1) a set of tensor equations involving only the components of the metric tensor and their derivatives; (2) the requirement that a certain matrix be semi-definite of rank one, the elements of which are polynomials in the components of the metric tensor and their derivatives; and (3) the requirement that the mean curvature of the space be different from zero. All results are proved for a simply connected neighborhood which is covered by a single coordinate system. (Received July 2, 1936.)

## 311. Professor C. C. Camp: *Expansions involving a system of differential equations in which the coefficients of the parameters change sign.*

For the single equation $X'+\lambda a(x)X=0$ and the boundary conditions $X(-1)=X(1)=0$ one treats by means of the lemma:

$\lim_{|s|\to\infty}\int_c\left[e^{2hz}dz/2\pi iz(e^{2z}-1)=\lim_{|s|\to\infty}\int_c[e^{-2hz}dz/2\pi iz(e^{-2z}-1)\right]=h-\frac{1}{2}$

when $2h$ is an integer, otherwise zero, the case in which $a(x)$ is real, integrable, and has a countable number of zeros but $\int_{-1}^{1}a(x)dx=2A_1\neq 0$. The points at which $a(x)=0$ are not significant but if $\int_{-1}^{\xi_i}a(x)dx=k_iA_1$, $k_i=0,\ \pm 1,\ \pm 2,\cdots$, then the expansion of $f(x)$, consisting of a finite number of pieces, each real, continuous and possessing a continuous derivative, will not be affected unless $f$ is discontinuous at one of the points $\xi_i$. In such a case the expansion will converge to the usual mean value plus terms each consisting of a linear combination of the $k$'s multiplied by $f(\xi_i+0)-f(\xi_i-0)$. This reasoning is extended to a system of $p$ linear equations in $p$ parameters. (Received July 2, 1936.)

312. Professor Leonard Carlitz: *On generalized Gauss sums.*

Let $\nu(A, M)$ denote the coefficient of $x^{k-1}$ in the reduced form of $A \pmod M$, where $A, B, \cdots, M$ are polynomials in an indeterminate $x$ with coefficients taken (mod $m$), $m>0$, and $M$ of degree $k$. Then the sums in question are of the form $\sum_A e(A^2H, M)$, summed over all $A \pmod M$, where $e(A, M) = \exp\{2\pi i\nu(A, M)/m\}$. These sums are evaluated by a very elementary method. Applications are made to a generalized theorem of quadratic reciprocity. (Received July 2, 1936.)

313. Professor Leonard Carlitz: *On the "singular series" for sums of squares of polynomials.*

By using the generalized Gauss sums described in the previous abstract, the "singular series" for the number of representations of a polynomial as a sum of squares is derived in an elementary way. The construction holds for the more general problem of determining the number of solutions of $M = A_1X_1^2 + \cdots + A_sX_s^2$, where $M, A_j$ are assigned polynomials in a single indeterminate with coefficients in a finite field. Certain related problems may be treated by the same methods. (Received July 2, 1936.)

314. Professor L. W. Griffiths: *Representation by generalized polygonal numbers.*

In my paper in the American Journal of Mathematics, vol. 55 (1933), pp. 102–110, there is a summary of the problem of representation of positive integers by generalized polygonal numbers as a generalization of representation by squares. Necessary conditions that the integers $0, 1, \cdots, 34m-16$ should be represented are obtained. Here $m$ is the positive integer defining the generalized polygonal numbers. In this paper, if the coefficients satisfy these necessary conditions, there is exhibited a positive integer $M$, depending only on $m$ and the coefficients, such that every integer greater than $M$ is represented. (Received July 3, 1936.)

315. Professor T. R. Hollcroft: *The binet of quadrics in $S_3$.*

Linear systems of quadrics in $S_3$ of dimension 1, 2, and 3 (pencils, nets and webs, respectively) have been treated extensively in the last half century, but nothing has been done specifically for a dimension greater than three. In the present paper, a linear $\infty 4$ system of quadrics is investigated. The name "binet" is suggested for such a system because and only because the dimension of the system is twice that of a net. In no other sense whatever is the binet a "double net." The properties of the binet are derived by considering the (1, 1) correspondence between the quadrics of the binet and the primes of $S_4$ and by studying the properties of the branch-point primal associated with this correspondence. The jacobians of the $\infty 4$ webs of the binet form another binet with a basis curve $\phi$ of order 10 and genus 11. The locus of axes of composite quadrics of the binet is a ruled surface of order 10 and index of irregularity 6 containing $\phi$ as a triple curve. The loci of contacts of quadrics of the binet are completely characterized. (Received July 2, 1936.)

316. Mr. H. M. MacNeille: *Extension of a multiplicative system to a Boolean ring.*

Consider a hypercomplex system, $H$, of unlimited rank, generated by an unrestricted set, $K$, of basis elements and an arbitrary ring, $R$, and such that each element of $H$ has but a finite number of non-zero components. If $K$ is an arbitrary multiplicative system (defined by the author, Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 47) and $R$ is the ring of integers modulo 2, then $H$ is a Boolean ring (defined by J. von Neumann and M. H. Stone, Fundamenta Mathematica, vol. 25 (1935), p. 353). If we suitably define an ideal, $M$, the quotient ring, $L=H/M$, is the smallest Boolean ring in which $K$ can be imbedded. If $K$ has a greatest element, then $L$ is a Boolean algebra and addition is the Boolean symmetric difference. The author has previously obtained essentially equivalent results by another method (loc. cit., p. 49). The present method is the more elegant and the more expedient when working with the symmetric difference (as Stone does, Transactions of this Society, to appear shortly). The former method is superior when working with the ordering relation and is necessary for the consideration of infinite operations. (Received July 2, 1936.)

317. Professor A. W. Tucker: *Branched and folded coverings.*

A *branched* covering of one $n$-complex by another is a generalization of the covering of a sphere by a Riemann surface; *folding* is an added complication quite well described by the intuitive meaning of the world. Through rudimentary combinatorial methods two formulas are developed which relate the Euler characteristics of the two complexes concerned in the covering with those of the subcomplexes about which branching and folding occur. One formula takes account of orientation; the other does not. (Received July 3, 1936.)

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